

MATH 142A: Introduction to Analysis

www.math.ucsd.edu/~ynemish/teaching/142a

Today: Sequences and their limits

> Q&A: January 15

Next: Ross § 9

Week 2:

- Quiz 1 (Wednesday, January 13) - Lectures 1-2
- homework 1 (due Friday, January 15)

Symbols $+\infty$ and $-\infty$

Often it is convenient to work with $\mathbb{R} \cup \{+\infty, -\infty\}$

Extend \leq to this set using rules:

- $\forall x \in \mathbb{R} \quad -\infty \leq x \leq +\infty$
- $-\infty \leq +\infty, -\infty \leq -\infty, +\infty \leq +\infty$

Use $\pm\infty$ to denote unbounded intervals

$$[a, +\infty) := \{x \in \mathbb{R} : x \geq a\}, \quad (a, +\infty) := \{x \in \mathbb{R} : x > a\}$$

$$(-\infty, b] := \{x \in \mathbb{R} : x \leq b\}, \quad (-\infty, b) := \{x \in \mathbb{R} : x < b\}$$

$$(-\infty, +\infty) := \mathbb{R}$$

We define

- $\sup S = +\infty$ if S is not bounded above
- $\inf S = -\infty$ if S is not bounded below

$$\sup \mathbb{N} = +\infty, \quad \inf \mathbb{Z} = -\infty$$

Sequences

Function, mapping: Let X and Y be two sets. We say that there is a function defined on X with values in Y , if via some rule f we associate to each element $x \in X$ an (one) element $y \in Y$. We write $f: X \rightarrow Y, x \mapsto y$ ($y = f(x)$)

X is called the domain of definition of the function,

$y = f(x)$ is called the image of x . E.g.: $f: [0,1) \rightarrow [0,1), x \mapsto x^2$

Def (Sequence) A function $f: \mathbb{N} \rightarrow X \subset \mathbb{R}$, whose domain of definition is the set of natural numbers, is called a **sequence**.

$f(1), f(2), f(3), f(4), \dots$

Notation: $(s_1, s_2, s_3, s_4, \dots) = (s_n)_{n=1}^{\infty}$

$f(m), f(m+1), f(m+2), \dots$

$(s_n)_{n \geq 1}, (s_n)_{n \in \mathbb{N}}, \{s_1, s_2, s_3, \dots\}, \{s_n\}_{n=1}^{\infty}$

$f(m-1+1), f(m-1+2), f(m-1+3), \dots$

$(s_m, s_{m+1}, s_{m+2}, \dots) = (s_n)_{n=m}^{\infty} = (s_n)_{n \geq m}$

$g(n) := f(m-1+n)$

Examples of sequences

• $(a_n)_{n=1}^{\infty}$, $a_n = 0$

$$(a_n)_{n=1}^{\infty} = (0, 0, 0, 0, \dots)$$

• $(a_n)_{n=1}^{\infty}$, $a_n = n$

$$(a_n)_{n=1}^{\infty} = (1, 2, 3, 4, \dots)$$

• $(a_n)_{n=1}^{\infty}$, $a_n = \frac{1}{n}$

$$(a_n)_{n=1}^{\infty} = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$$

• $b_n = \frac{1}{2^n}$, $n \in \{0, 1, 2, 3, \dots\}$

$$(b_n)_{n=0}^{\infty} = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots)$$

• $\sin\left(\frac{n\pi}{2}\right)$, $n \in \mathbb{N}$

$$(1, 0, -1, 0, 1, 0, \dots)$$

• $b_n = \left(1 + \frac{1}{n}\right)^n$, $n \in \mathbb{N}$

$$(b_n)_{n=1}^{\infty} = (2, 2.25, 2.3704, 2.4414, 2.5216, \dots)$$

• $a_n = n^2 \cdot \sin\left(\frac{1}{n^2}\right)$

$$(a_n)_{n=1}^{\infty} = (0.84, 0.98, 0.997, 0.9993, 0.9997, \dots)$$

Convergence

Def 7.1. A sequence (s_n) of real numbers is said to **converge** to the real number s if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n > N (|s_n - s| < \varepsilon)$$

Notation: $\lim_{n \rightarrow \infty} s_n = s$ or $s_n \rightarrow s, n \rightarrow \infty$

Def A sequence that does not converge is said to **diverge**.
convergent / divergent

Remark $N \in \mathbb{N}$ in the definition depends on ε .

$$s_n = \frac{n+1}{n}, n \in \mathbb{N}, \lim_{n \rightarrow \infty} s_n = 1, |s_n - 1| = \left| \frac{n+1}{n} - 1 \right| = \frac{1}{n}$$

$$|s_n - 1| < 1 \text{ for all } n > 1$$

$$|s_n - 1| < 0.1 \text{ for all } n > 10$$

$$|s_n - 1| < 0.01 \text{ for all } n > 100 \text{ (also for all } n > 101, n > 1000)$$

Examples of sequences

• $(a_n)_{n=1}^{\infty}$, $a_n = 0$

$(a_n)_{n=1}^{\infty} = (0, 0, 0, 0, \dots)$ $\lim_{n \rightarrow \infty} a_n = 0$

• $(a_n)_{n=1}^{\infty}$, $a_n = n$

$(a_n)_{n=1}^{\infty} = (1, 2, 3, 4, \dots)$ diverges

• $(a_n)_{n=1}^{\infty}$, $a_n = \frac{1}{n}$

$(a_n)_{n=1}^{\infty} = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$ $\lim_{n \rightarrow \infty} a_n = 0$

• $b_n = \frac{1}{2^n}$, $n \in \{0, 1, 2, \dots\}$

$(b_n)_{n=0}^{\infty} = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots)$ $\lim_{n \rightarrow \infty} a_n = 0$

• $\sin\left(\frac{n\pi}{2}\right)$, $n \in \mathbb{N}$

$(1, 0, -1, 0, 1, \dots)$ diverges

• $b_n = \left(1 + \frac{1}{n}\right)^n$, $n \in \mathbb{N}$

$(b_n)_{n=1}^{\infty} = (2, 2.25, 2.3704, 2.4414, 2.5216, \dots)$

$\lim_{n \rightarrow \infty} b_n = e$

• $a_n = n^2 \sin\left(\frac{1}{n^2}\right)$, $n \in \mathbb{N}$

$(a_n)_{n=1}^{\infty} = (0.84, 0.98, 0.997, 0.9993, 0.9997, \dots)$

$\lim_{n \rightarrow \infty} n^2 \sin\left(\frac{1}{n^2}\right) = 1$

Uniqueness of limit

Prop. Let $(s_n)_{n=1}^{\infty}$ be a convergent sequence. Then

$$\lim_{n \rightarrow \infty} s_n = s \wedge \lim_{n \rightarrow \infty} s_n = t \Rightarrow s = t$$

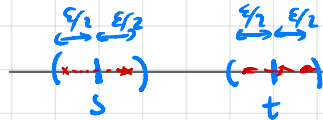
Proof. Fix $\varepsilon > 0$. Then

$$\textcircled{1} \quad \lim_{n \rightarrow \infty} s_n = s \Rightarrow \exists N_1 \forall n > N_1 \quad |s_n - s| < \frac{\varepsilon}{2}$$

$$\textcircled{2} \quad \lim_{n \rightarrow \infty} s_n = t \Rightarrow \exists N_2 \forall n > N_2 \quad |s_n - t| < \frac{\varepsilon}{2}$$

$$\textcircled{3} \quad \textcircled{1} \text{ and } \textcircled{2} \Rightarrow \forall n > \max\{N_1, N_2\}$$

$$\begin{array}{l} \text{Tr. Ineq.} \\ |s - t| \leq |s_n - s| + |s_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{array}$$



$$\Rightarrow |s - t| < \varepsilon \text{ for any } \varepsilon > 0 \Rightarrow |s - t| = 0 \Rightarrow s = t$$



Example

Let $p \in \mathbb{Z}$. Then $\lim_{n \rightarrow \infty} n^p = \begin{cases} 0, & p < 0 & \text{(a)} \\ 1, & p = 0 & \text{(b)} \\ \text{diverges}, & p > 0 & \text{(c)} \end{cases}$

Proof (b) $n^0 = 1 \Rightarrow \forall \varepsilon > 0 \quad \forall n \in \mathbb{N} \quad |n^0 - 1| = 0 < \varepsilon$. (take $N = 1$)

(c) Suppose $\exists s \in \mathbb{R}$ s.t. $\lim_{n \rightarrow \infty} n^p = s$. Then (take $\varepsilon = 1$)

$$\exists N \in \mathbb{N} \quad \forall n > N \quad |n^p - s| < 1$$

$$\Rightarrow \forall n > N \quad n^p < 1 + s \Rightarrow \forall n > N \quad n < \sqrt[p]{1+s} \quad \text{(contradiction AP)}$$

$\Rightarrow n^p$ is divergent

(a) Fix $\varepsilon > 0$, denote $q = -p \in \mathbb{N}$. $\left\{ \text{find } N \text{ s.t. } \forall n > N \quad \frac{1}{n^q} < \varepsilon \Leftrightarrow n^q > \frac{1}{\varepsilon} \right.$

Take $N = \left\lceil \sqrt[q]{\frac{1}{\varepsilon}} \right\rceil$. Then for $n > N$

$$n^q > \frac{1}{\varepsilon} \Leftrightarrow \frac{1}{n^q} < \varepsilon \Leftrightarrow n^p < \varepsilon \Leftrightarrow |n^p - 0| < \varepsilon$$

Example

$$\lim_{n \rightarrow \infty} \frac{5n^4 - n - 10}{7n^4 - n^2} = \frac{5}{7}$$

$$\frac{5n^4 - n - 10}{7n^4 - n^2} = \frac{\cancel{n^4} \left(5 - \frac{1}{n^3} - \frac{10}{n^4}\right)}{\cancel{n^4} \left(7 - \frac{1}{n^2}\right)}$$

$$\left| \frac{5n^4 - n - 10}{7n^4 - n^2} - \frac{5}{7} \right| = \left| \frac{\cancel{35n^4} - 7n - 70 - \cancel{35n^4} - 5n^2}{49n^4 - 7n^2} \right| = \left| \frac{-5n^2 - 7n - 70}{49n^4 - 7n^2} \right| < \varepsilon$$

$$\forall n > 10 \quad \underbrace{5n^2}_{\downarrow n^2} + \underbrace{7n}_{\downarrow n^2} + 70 < 7n^2 \Rightarrow \left| \frac{-5n^2 - 7n - 70}{7n^2(7n^2 - 1)} \right| < \frac{\cancel{7n^2}}{\cancel{7n^2}(7n^2 - 1)} = \frac{1}{7n^2 - 1} < \frac{1}{n^2} < \varepsilon$$

$$N = \max\{10, \lceil \frac{1}{\varepsilon} \rceil\}$$

Proof Fix $\varepsilon > 0$. Then $\forall n > \max\{10, \lceil \frac{1}{\varepsilon} \rceil\} = N$

$$\left| \frac{5n^4 - n - 10}{7n^4 - n^2} - \frac{5}{7} \right| = \left| \frac{-5n^2 - 7n - 10}{7n^2(7n^2 - 1)} \right| < \frac{7n^2}{7n^2(7n^2 - 1)} < \frac{1}{6n^2} < \frac{1}{n^2},$$

$$\text{and } n > \lceil \frac{1}{\varepsilon} \rceil \Rightarrow n > \frac{1}{\varepsilon} \Rightarrow n^2 > \frac{1}{\varepsilon} \Rightarrow \frac{1}{n^2} < \varepsilon \quad \blacksquare$$