MATH 142A: Introduction to Analysis

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Today: Limit theorems for sequences

> Q&A: January 15, 20

Next: Ross § 9

Week 2:

- homework 1 (due Friday, January 15)

Last time

Def 7.1. A sequence (s_n) of real numbers is said to converge to the real number s if

$$\forall \varepsilon>0 \; \exists N\in\mathbb{N} \; \forall n>N \quad (|s_n - s| < \varepsilon)$$

$$\lim_{n\to\infty} s_n = s \quad , \quad s_n \to s \; , \; n\to\infty$$

Example

Let $p\in\mathbb{Z}$. Then

$$\lim_{n\to\infty} n^p = \begin{cases} 0, & p<0 \quad \text{(a)} \\ 1, & p=0 \quad \text{(b)} \\ \text{diverges}, & p>0 \quad \text{(c)} \end{cases}$$

$\frac{1}{n^q}$, $q>0$

Example

$$\lim_{n\to\infty} \frac{5n^4 - n - 10}{7n^4 - n^2} = \frac{5}{7}$$

Convergent sequences are bounded

Def (Bounded sequence).

A sequence (s_n) is said to be bounded if the set $\{s_n : n \in \mathbb{N}\}$ is bounded (i.e., $\exists\, M > 0\ \forall n \in \mathbb{N}\ |s_n| < M$)

Thm 9.1 Let (s_n) be convergent. Then (s_n) is bounded

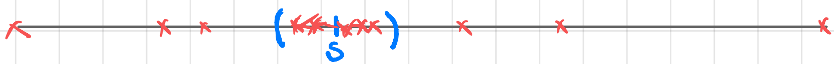
Proof. Let $s = \lim_{n\to\infty} s_n$, $s \in \mathbb{R}$. Then by Def. 7.1 $(\varepsilon = 1)$

$$\exists\, N\ \forall n > N \quad |s_n - s| < 1$$

By the triangle inequality, $|s_n| \le |s| + |s_n - s|$

therefore $\forall n > N \quad |s_n| < |s| + 1$

Take $M = \max\{|s_1|, |s_2|, \dots, |s_N|, |s|+1\}$. Then $\forall n \in \mathbb{N}\ |s_n| \le M$ ∎



Multiplying covergent sequence by a scalar

Thm 9.2

Let (s_n) be convergent, $\lim_{n\to\infty} s_n = s \in \mathbb{R}$, and let $k \in \mathbb{R}$.

Then $\lim_{n\to\infty} k\cdot s_n = k\cdot s$ (i.e. $\lim_{n\to\infty} k\cdot s_n = k\cdot \lim_{n\to\infty} s_n$)

Proof. If $k=0$, then $\forall \varepsilon>0\ \forall n\in\mathbb{N}\ |k\cdot s_n| = 0 < \varepsilon$, and thus $\lim_{n\to\infty} k\cdot s_n = 0 = 0\cdot s$

Suppose $k\neq 0$. Fix $\varepsilon>0$

$\exists N\in\mathbb{N}\quad \forall n>N$

$|ks_n - ks| < \varepsilon \ (\Leftrightarrow |k|\,|s_n - s| < \varepsilon \Leftrightarrow |s_n - s| < \frac{\varepsilon}{|k|}$

$\lim_{n\to\infty} s_n = s \Rightarrow \exists N\in\mathbb{N}\ \forall n>N\ |s_n - s| < \frac{\varepsilon}{|k|}$ (Def 7.1 with $\frac{\varepsilon}{|k|}$)

Then $\forall n>N\quad |ks_n - ks| = |k|\,|s_n - s| < |k|\cdot\frac{\varepsilon}{|k|} = \varepsilon$ ∎

Example

- $\lim_{n\to\infty}\frac{10}{n^4} = \lim_{n\to\infty} 10\cdot\frac{1}{n^4} = 10\cdot\lim_{n\to\infty}\frac{1}{n^4} = 10\cdot 0 = 0$
- $\forall k\in\mathbb{R}$, $\lim_{n\to\infty} k = \lim_{n\to\infty} k\cdot 1 = k\lim_{n\to\infty} 1 = k\cdot 1 = k$

Limit of a sum

Thm 9.3 Let (s_n) and (t_n) be two convergent sequences.

If $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$, then $\lim_{n\to\infty}(s_n + t_n) = s+t$ $\left(\lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n\right)$

Proof. Fix $\varepsilon > 0$.

$$\left\{ \begin{array}{l} \exists N \in \mathbb{N} \ \forall n > N \quad |s_n + t_n - (s+t)| < \varepsilon \\ |s_n - s + t_n - t| \overset{\text{Tr. Ineq.}}{\leq} |s_n - s| + |t_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{array} \right.$$

$\lim_{n\to\infty} s_n = s \Rightarrow \exists N_1 \ \forall n > N_1 \quad |s_n - s| < \frac{\varepsilon}{2}$

$\lim_{n\to\infty} t_n = t \Rightarrow \exists N_2 \ \forall n > N_2 \quad |t_n - t| < \frac{\varepsilon}{2}$

Then $\forall n > N := \max\{N_1, N_2\}$ $|s_n + t_n - (s+t)| \leq |s_n - s| + |t_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ ∎

Corollary $(s_n), (t_n)$ convergent $\Rightarrow \lim_{n\to\infty}(s_n - t_n) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} t_n$

Example $\lim_{n\to\infty}\left(5 - \frac{1}{n^3} - \frac{10}{n^4}\right) = \lim_{n\to\infty} 5 - \lim_{n\to\infty}\frac{1}{n^3} - \lim_{n\to\infty}\frac{10}{n^4} = 5 - 0 - 0 = 5$

Limit of a product

Thm 9.4 Let (s_n) and (t_n) be convergent, $\lim_{n\to\infty} s_n = s \in \mathbb{R}$, $\lim_{n\to\infty} t_n = t \in \mathbb{R}$.

Then $\lim_{n\to\infty}(s_n \cdot t_n) = s \cdot t = \left(\lim_{n\to\infty} s_n\right)\left(\lim_{n\to\infty} t_n\right)$

Proof Fix $\varepsilon > 0$.

$\exists N \ \forall n > N \ |s_n t_n - st| < \varepsilon$, can control $|s_n - s|$, $|t_n - t|$

$$|s_n t_n - st| = |s_n t_n - s_n t + s_n t - st| \le \underbrace{|s_n||t_n - t|}_{\overset{?}{<} \varepsilon/2} + \underbrace{|t||s_n - s|}_{\overset{?}{<} \varepsilon/2}$$

Thm 9.1

$\exists M > 0 : |s_n| \le M \Rightarrow |s_n||t_n - t| \le M|t_n - t| < \frac{\varepsilon}{2}$

$|t| < |t| + 1 \Rightarrow |t||s_n - s| < (|t| + 1)|s_n - s| < \frac{\varepsilon}{2}$

$\exists N_1 \in \mathbb{N} \ \forall n > N_1 \ |s_n - s| < \frac{\varepsilon}{2(|t|+1)}$

$\exists N_2 \in \mathbb{N} \ \forall n > N_2 \ |t_n - t| < \frac{\varepsilon}{2M}$

$\Longrightarrow$ $\forall n > \max\{N_1, N_2\} =: N$

$|s_n t_n - st| \le M|t_n - t| + |t||s_n - s| < \varepsilon$ ∎

Example

$$\lim_{n\to\infty}\left(5 - \frac{1}{n^3} - \frac{10}{n^4}\right)\left(7 - \frac{1}{n^2}\right) = \lim_{n\to\infty}\left(5 - \frac{1}{n^3} - \frac{10}{n^4}\right)\lim_{n\to\infty}\left(7 - \frac{1}{n^2}\right) = 5 \cdot 7 = 35$$

$\downarrow$ 5 ($n \to \infty$), $\downarrow$ 7

Limit of a sequence of reciprocals

Thm 9.5 Let (s_n) be a convergent sequence, $\lim_{n\to\infty} s_n = s$ such that $(\forall n \in \mathbb{N}\ (s_n \neq 0)) \wedge (s \neq 0)$. Then

$$\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s} = \frac{1}{\lim_{n\to\infty} s_n}$$

Proof. Fix $\varepsilon > 0$.

$$\exists N \in \mathbb{N}\ \forall n > N \quad \left|\frac{1}{s_n} - \frac{1}{s}\right| < \varepsilon$$

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s_n - s}{s_n s}\right| = \frac{|s_n - s|}{|s_n||s|} \overset{?}{<} \varepsilon$$

If $|s_n| \geq m > 0$, then $\left|\frac{1}{s_n} - \frac{1}{s}\right| \leq \frac{1}{|s| m} |s_n - s|$

$$|s_n - s| < |s| m \varepsilon \Rightarrow \left|\frac{1}{s_n} - \frac{1}{s}\right| < \varepsilon$$

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① $\exists m > 0 \ \inf\{|s_n| : n \in \mathbb{N}\} \geq m$. Proof①: $\exists N_1\ \forall n > N_1\ |s_n - s| < \frac{|s|}{2}$. Then

$$\forall n > N_1 \quad |s_n| \overset{\text{Tr. Ineq.}}{\geq} |s| - |s_n - s| > |s| - \frac{|s|}{2} = \frac{|s|}{2}.$$ Take $m = \min\{|s_1|, |s_2|, \dots, |s_N|, \frac{|s|}{2}\} > 0$

Limit of a fraction of two convergent sequences

② $\exists N_2 \in \mathbb{N} \quad \forall n > N_2 \quad |s_n - s| < |s| \cdot m \cdot \varepsilon$

$$\forall n > N_2 \qquad \left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s_n - s}{s_n s}\right| \leq \frac{|s_n - s|}{|s|\, m} < \frac{\varepsilon \cdot |s| \cdot m}{|s| \cdot m} = \varepsilon$$

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Thm 9.6.

Let $(s_n), (t_n)$ be two convergent sequences, $\lim\limits_{n\to\infty} s_n = s$, $\lim\limits_{n\to\infty} t_n = t$, $\forall n \in \mathbb{N} \quad s_n \neq 0$, $s \neq 0$. Then

$$\lim_{n\to\infty} \frac{t_n}{s_n} = \frac{t}{s} = \frac{\lim\limits_{n\to\infty} t_n}{\lim\limits_{n\to\infty} s_n}.$$

Proof

$$\lim_{n\to\infty} \frac{t_n}{s_n} = \lim_{n\to\infty} t_n \cdot \frac{1}{s_n} \overset{\substack{\text{Thm 9.5} \\ \text{Thm 9.4}}}{=} \lim_{n\to\infty} t_n \cdot \lim_{n\to\infty} \frac{1}{s_n} \overset{\text{Thm 9.5}}{=} t \cdot \frac{1}{s} = \frac{t}{s}$$

■

Examples

1) $$\lim_{n\to\infty} \frac{5n^4-n-10}{7n^4-n^2} = \frac{5}{7}$$

$$\lim_{n\to\infty} \frac{5n^4-n-10}{7n^4-n^2} \overset{\text{Thm 9.6}}{\neq} \frac{\lim_{n\to\infty}(5n^4-n-10) \rightarrow \text{diverges}}{\lim_{n\to\infty}(7n^4-n^2) \rightarrow \text{diverges}}$$

$$\parallel$$

$$\lim_{n\to\infty} \frac{\cancel{n^4}\left(5-\frac{1}{n^3}-\frac{10}{n^4}\right)}{\cancel{n^4}\left(7-\frac{1}{n^2}\right)} = \frac{\lim_{n\to\infty}\left(5-\frac{1}{n^3}-\frac{10}{n^4}\right)}{\lim_{n\to\infty}\left(7-\frac{1}{n^2}\right)} = \frac{5}{7}$$

2) $$\lim_{n\to\infty} \frac{5n^4-n-10}{7n^5-n^2} = \lim_{n\to\infty} \frac{\cancel{n^4}\left(5-\frac{1}{n^2}-\frac{10}{n^4}\right)}{n^5\left(7-\frac{1}{n^3}\right)} = \lim_{n\to\infty} \frac{5-\frac{1}{n^2}-\frac{10}{n^4}}{n\left(7-\frac{1}{n^3}\right)}$$

$$= \lim_{n\to\infty} \frac{1}{n}\cdot\frac{5-\frac{1}{n^2}-\frac{10}{n^4}}{7-\frac{1}{n^3}} = \lim_{n\to\infty}\frac{1}{n}\ \lim_{n\to\infty}\frac{5-\frac{1}{n^2}-\frac{10}{n^4}}{7-\frac{1}{n^3}} = 0\cdot\frac{5}{7} = 0$$

Examples

3) $$\lim_{n\to\infty} \frac{5n^5 - n - 10}{7n^4 - n^2} =$$

$$= \frac{n^5\left(5 - \frac{1}{n^4} - \frac{10}{n^5}\right)}{n^4\left(7 - \frac{1}{n^2}\right)} = n\,\frac{5 - \frac{1}{n^4} - \frac{10}{n^4}}{7 - \frac{1}{n^2}}$$