MATH 142A: Introduction to Analysis

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Today: Limit theorems for sequences

> Q&A: January 15, 20

Next: Ross § 9

Week 2:

- homework 1 (due Friday, January 15)

Last time

Def 7.1. A sequence (s_n) of real numbers is said to converge to the real number s if

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n > N \quad (|s_n - s| < \varepsilon)$$

$$\lim_{n\to\infty} s_n = s \quad , \quad s_n \to s \, , \; n \to \infty$$

Example

Let $p \in \mathbb{Z}$. Then

$$\lim_{n\to\infty} n^p = \begin{cases} 0, & p < 0 & \text{(a)} \\ 1, & p = 0 & \text{(b)} \\ \text{diverges}, & p > 0 & \text{(c)} \end{cases} \qquad \frac{1}{n^q}, \; q > 0$$

Example

$$\lim_{n\to\infty} \frac{5n^4 - n - 10}{7n^4 - n^2} = \frac{5}{7}$$

Convergent sequences are bounded

Def (Bounded sequence).

A sequence (s_n) is said to be if

the set $\{s_n : n \in \mathbb{N}\}$ is bounded (i.e.,

Thm 9.1

Let (s_n) be convergent. Then

Proof. Let $s = \lim_{n\to\infty} s_n$, $s \in \mathbb{R}$. Then by Def. 7.1

By the triangle inequality,

therefore $\forall\, n > N$

Take $M =$ Then

Multiplying covergent sequence by a scalar

Thm 9.2 Let (s_n) be convergent, $\lim_{n\to\infty} s_n = s \in \mathbb{R}$, and let $k \in \mathbb{R}$.

Then (i.e.)

Proof. If $k=0$, then , and thus

Suppose $k \neq 0$.

$\lim_{n\to\infty} s_n = s \implies$

Then $\forall n > N$

Example

- $\lim_{n\to\infty} \frac{10}{n^4} =$
- $\forall\ k \in \mathbb{R}$,

Limit of a sum

Thm 9.3 Let (s_n) and (t_n) be two convergent sequences.

If $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$, then

Proof. Fix $\varepsilon > 0$.

$\lim_{n\to\infty} s_n = s \Rightarrow$

$\lim_{n\to\infty} t_n = t \Rightarrow$

Then

Corollary $(s_n), (t_n)$ convergent $\Rightarrow$

Example $\lim_{n\to\infty} \left(5 - \frac{1}{n^3} - \frac{10}{n^4}\right) =$

Limit of a product

Thm 9.4 Let (s_n) and (t_n) be convergent, $\lim_{n\to\infty} s_n = s \in \mathbb{R}$, $\lim_{n\to\infty} t_n = t \in \mathbb{R}$.

Then

Proof Fix $\varepsilon > 0$.

Example

$$\lim_{n\to\infty} \left(5 - \frac{1}{n^3} - \frac{10}{n^4}\right)\left(7 - \frac{1}{n^2}\right) =$$

Limit of a sequence of reciprocals

Thm 9.5 Let (s_n) be a convergent sequence, $\lim_{n\to\infty} s_n = s$

such that

Then

Proof. Fix $\varepsilon > 0$.

①

Limit of a fraction of two convergent sequences

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Thm 9.6.

Let $(s_n), (t_n)$ be two convergent sequences, $\lim_{n\to\infty} s_n = s$, $\lim_{n\to\infty} t_n = t$,

$\forall n \in \mathbb{N}$ $s_n \neq 0$, $s \neq 0$. Then

Proof

Examples

1) $$\lim_{n\to\infty} \frac{5n^4-n-10}{7n^4-n^2} = \frac{5}{7}$$

$$\lim_{n\to\infty} \frac{5n^4-n-10}{7n^4-n^2} \overset{\text{Thm 9.6}}{=}$$

2) $$\lim_{n\to\infty} \frac{5n^4-n-10}{7n^5-n^2} =$$

Examples

3) $$\lim_{n\to\infty} \frac{5n^5 - n - 10}{7n^4 - n^2} =$$