MATH 142A: Introduction to Analysis

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Today: Monotone sequences

> Q&A: January 22

Next: Ross § 10

Week 3:

- Homework 2 (due Friday, January 22)
- Midterm 1 on Wednesday, January 27 (lectures 1-7)
- Regrades for HW1: Mon, Jan 25 - Tue, Jan 26 (PST) on Gradescope

Monotone sequences

Def 10.1 A sequence (s_n) is called

an increasing sequence if $\forall n \quad s_n \leq s_{n+1}$

a decreasing sequence if $\forall n \quad s_n \geq s_{n+1}$

a monotone / monotonic sequence if it is increasing or decreasing

Examples

$a_n = 0$ increasing and decreasing

$b_n = n$ increasing

$c_n = -n$ decreasing

$d_n = \frac{1}{n}$ decreasing

$e_n = \frac{(-1)^n}{n}$ not monotonic

$f_n = \frac{n}{1+n}$ increasing

$$\frac{n+1}{n+2} - \frac{n}{n+1} = \frac{1}{(n+2)(n+1)} > 0$$

$g_n = n^2 - 4n$ not monotone

$-3, -4, -3, \dots$

$h_n = \left(1 + \frac{1}{n}\right)^n$ increasing

Bounded monotone sequences converge

Thm 10.2 All bounded monotonic sequences converge.

Proof Let (s_n) be a bounded increasing sequence.

Denote $S := \{ s_n : n \in \mathbb{N} \}$. Then

(s_n) bounded $\Rightarrow$ $u := \sup S \in \mathbb{R}$

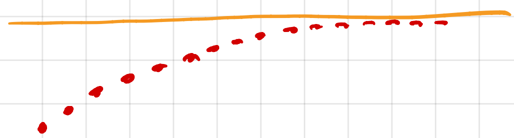
Fix $\varepsilon > 0$. Then

① $u = \sup S \Rightarrow \forall n \quad s_n \leq u < u + \varepsilon$

② $u = \sup S \Rightarrow \exists N \quad s_N > u - \varepsilon$

(s_n) is increasing $\Rightarrow \forall n > N \quad u - \varepsilon < s_N \leq s_n$

① + ② imply that $\forall n > N \quad u - \varepsilon < s_n < u + \varepsilon$ ∎



Show $\lim_{n \to \infty} s_n = \sup S$

$\forall \varepsilon > 0 \; \exists N \; \forall n > N$

$|s_n - u| < \varepsilon \iff u - \varepsilon < s_n < u + \varepsilon$

Important example: the number e

Sequence $a_n = \left(1+\frac{1}{n}\right)^n$ is convergent.

Proof. ① Sequence $b_n := \left(1+\frac{1}{n}\right)^{n+1}$ is decreasing

$$\forall n \geq 2 \quad \frac{b_{n-1}}{b_n} = \frac{\left(1+\frac{1}{n-1}\right)^n}{\left(1+\frac{1}{n}\right)^{n+1}} = \frac{n}{n+1}\left(\frac{n}{n-1}\cdot\frac{n}{n+1}\right)^n = \frac{n}{n+1}\left(\frac{n^2}{n^2-1}\right)^n = \frac{n}{n+1}\left(1+\frac{1}{n^2-1}\right)^n$$

Now use Bernoulli's inequality: $\left(1+\frac{1}{n^2-1}\right)^n > 1+n\cdot\frac{1}{n^2-1} > 1+\frac{n}{n^2} > 1+\frac{1}{n}$, so

$$\frac{b_{n-1}}{b_n} > \frac{n}{n+1}\cdot\frac{n+1}{n} = 1 \Rightarrow \forall n \geq 2 \quad b_{n-1} > b_n, \ (b_n) \text{ is decreasing.}$$

② Sequence b_n is bounded below: $\forall n \ \ b_n > 0$

③ ① + ② + Thm. 10.2 $\Rightarrow$ sequence $(b_n)_{n=1}^{\infty}$ converges

④ $$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}\cdot\frac{n}{n+1} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} =: e$$

$\blacksquare$

Example

Consider the sequence $(a_n)_{n=1}^{\infty}$ given by $a_1 = \sqrt{5}$, $\overbrace{a_{n+1} = \sqrt{5+a_n}}^{a_{n+1}^2 = 5+a_n}$

Is (a_n) convergent? If yes, what is the limit?

① (a_n) is monotonic ✓ ① $\forall n \;\; a_n > 0$

$$a_{n+1} - a_n = \sqrt{5+a_n} - \sqrt{5+a_{n-1}} = \frac{5+a_n-(5+a_{n-1})}{\sqrt{5+a_n}+\sqrt{5+a_{n-1}}} = \frac{a_n - a_{n-1}}{\underbrace{\sqrt{5+a_n}+\sqrt{5+a_{n-1}}}_{>0}}$$

$a_2 - a_1 = \sqrt{5+\sqrt{5}} - \sqrt{5} > 0$; if $a_n - a_{n-1} > 0$, then $a_{n+1} - a_n > 0$

Principle of math induction $\Rightarrow \forall n \;\; a_{n+1} - a_n > 0$, (a_n) is increasing

② (a_n) is bounded above ✓

$$a_{n+1} = \sqrt{5+a_n} = \sqrt{5\left(1+\frac{a_n}{5}\right)} \overset{\text{AM-GM}}{\leq} \frac{5+1+\frac{a_n}{5}}{2} = 3 + \frac{a_n}{10} \leq 3 + \frac{1}{10}\left(3 + \frac{a_{n-1}}{10}\right)$$

$$\leq 3 + \frac{3}{10} + \frac{3}{10^2} + \dots + \frac{3}{10^{n+1}} = 3\left(1 + \frac{1}{10} + \frac{1}{10^2} + \dots + \frac{1}{10^{n+1}}\right) < 3 \cdot 2 = 6$$

① + ② + Thm. 10.2 $\Rightarrow \lim\limits_{n\to\infty} a_n = A \in \mathbb{R}$. Define $b_n := a_{n+1} \cdot a_{n+1} - a_n - 5 = 0$

$$\lim_{n\to\infty} b_n = 0 \overset{\text{Thm 9.4, 9.3}}{=} A^2 - A - 5 \Rightarrow A \in \left\{ \frac{1+\sqrt{21}}{2}, \overset{<0}{\frac{1-\sqrt{21}}{2}} \right\} \Rightarrow A = \frac{1+\sqrt{21}}{2} \;\; \blacksquare$$

Unbounded monotone sequences

Thm 10.4 (i) If (s_n) is unbounded and increasing, then $\lim_{n\to\infty} s_n = +\infty$

(ii) If (s_n) is unbounded and decreasing, then $\lim_{n\to\infty} s_n = -\infty$

Proof (i) Fix $M > 0$. (s_n) unbounded $\Rightarrow \exists N \in \mathbb{N} \quad s_N > M$

(s_n) increasing $\Rightarrow \forall n > N \quad s_n \geq s_N > M$

$\blacksquare$

Corollary 10.5 If (s_n) is a monotone sequence, then it has a limit

i.e., (s_n) converges or diverges to $+\infty$ or diverges to $-\infty$.

limsup and liminf

Let (s_n) be convergent, $\lim_{n\to\infty} s_n = s$. Then $\forall \varepsilon>0\ \exists N$

$$\forall n>N\ |s_n - s| < \varepsilon \iff \forall n>N\quad s-\varepsilon < s_n < s+\varepsilon \iff \begin{array}{l} \inf\{s_n : n>N\} \geq s-\varepsilon \\ \sup\{s_n : n>N\} \leq s+\varepsilon \end{array}$$

$$\lim_{n\to\infty} s_n = s \quad \text{iff} \quad \forall\varepsilon>0\ \exists N \quad s-\varepsilon \leq \underbrace{\inf\{s_n : n>N\}}_{= u_N} \leq \underbrace{\sup\{s_n : n>N\}}_{= v_N} \leq s+\varepsilon$$

$$\left.\begin{array}{l} (u_n)_{n=1}^{\infty} \text{ is increasing} \\ (v_n)_{n=1}^{\infty} \text{ is decreasing} \end{array}\right| \Rightarrow \left(\lim_{n\to\infty} s_n = s \Rightarrow \begin{array}{l} \lim_{n\to\infty} u_n = s \\ \lim_{n\to\infty} v_n = s \end{array} \right)$$

Def 10.6 Let (s_n) be a sequence. We define

$$\limsup_{n\to\infty} s_n = \overline{\lim_{n\to\infty}}\, s_n = \lim_{N\to\infty} \sup\{s_n : n>N\}$$

$$\liminf_{n\to\infty} s_n = \underline{\lim_{n\to\infty}}\, s_n := \lim_{N\to\infty} \inf\{s_n : n>N\}$$

If $\sup\{s_n : n\in\mathbb{N}\} = +\infty$, $\limsup_{n\to\infty} s_n = +\infty$; if $\inf\{s_n : n\in\mathbb{N}\} = -\infty$, $\liminf_{n\to\infty} s_n = -\infty$

limsup and liminf

Examples

1) $a_n = n$,

$$\forall N \quad \sup\{a_n : n > N\} = +\infty \Rightarrow \limsup_{n\to\infty} n = +\infty$$

$$\forall N \quad \inf\{a_n : n > N\} = N+1 \Rightarrow \liminf_{n\to\infty} n = +\infty$$

2) $b_n = \frac{1}{n}$,

$$\forall N \quad \sup\{b_n : n > N\} = \frac{1}{N+1} \Rightarrow \limsup_{n\to\infty} \frac{1}{n} = 0$$

$$\forall N \quad \inf\{b_n : n > N\} = 0 \Rightarrow \liminf_{n\to\infty} \frac{1}{n} = 0$$

3) $c_n = \frac{(-1)^n}{n}$

$$\forall N \quad \sup\{c_n : n > N\} \leq \frac{1}{N+1} \Rightarrow \limsup_{n\to\infty} \frac{(-1)^n}{n} \overset{=}{\leq} 0$$

$$\forall N \quad \inf\{c_n : n > N\} \geq -\frac{1}{N+1} \Rightarrow \liminf_{n\to\infty} \frac{(-1)^n}{n} \underset{=}{\geq} 0$$

4) $d_n = (-1)^n$

$$\forall N \quad \sup\{d_n : n > N\} = 1 \Rightarrow \limsup_{n\to\infty} (-1)^n = 1$$

$$\forall N \quad \inf\{d_n : n > N\} = -1 \Rightarrow \liminf_{n\to\infty} (-1)^n = -1$$

5) $e_n = n^{(-1)^n}$

$$\forall N \quad \sup\{e_n : n > N\} = +\infty \Rightarrow \limsup_{n\to\infty} n^{(-1)^n} = +\infty$$

$$\forall N \quad \inf\{e_n : n > N\} = 0 \Rightarrow \liminf_{n\to\infty} n^{(-1)^n} = 0$$

Convergence and limsup/liminf

Thm. 10.7 Let (s_n) be a sequence in $\mathbb{R}$, $s \in \mathbb{R}$ or $s \in \{+\infty, -\infty\}$.

Then (i) $\lim_{n\to\infty} s_n = s \implies \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n = s$

(ii) $\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = s \implies \lim_{n\to\infty} s_n = s$

Proof Denote $u_N = \inf\{s_n : n > N\}$, $v_N = \sup\{s_n : n > N\}$, $u = \liminf_{n\to\infty} s_n$, $v = \limsup_{n\to\infty} s_n$

(i) Three cases: $s = +\infty$, $s = -\infty$, $s \in \mathbb{R}$

$\boxed{s = +\infty}$ Fix $M > 0$. Then $\exists N \ \forall n > N \ s_n > 2M$, and this implies

that $\forall n > N \quad u_n \geq u_N \geq 2M > M \implies \lim_{N\to\infty} u_N = \liminf_{n\to\infty} s_n = +\infty$

On the other hand, $\lim_{n\to\infty} s_n = +\infty \implies \forall n \quad v_n = +\infty \implies \limsup_{n\to\infty} s_n = +\infty$

$\boxed{s \in \mathbb{R}}$ Fix $\varepsilon > 0$. Then $\exists N \ \forall n > N \ s - \frac{\varepsilon}{2} < s_n < s + \frac{\varepsilon}{2}$ Then

(1) $u_n \geq u_N \geq s - \frac{\varepsilon}{2} > s - \varepsilon \implies \forall n > N \quad u_n > s - \varepsilon$ (2) $v_n \leq v_N \leq s + \frac{\varepsilon}{2} < s + \varepsilon \implies \forall n > N \quad v_n < s + \varepsilon$

(3) $\forall n \quad u_n \leq v_n$ (1)+(2)+(3) $\implies \forall n > N \quad s - \varepsilon < u_n \leq v_n < s + \varepsilon$

Convergence and limsup/liminf

(ii) Three cases: $s = +\infty$, $s = -\infty$, $s \in \mathbb{R}$.

$\boxed{s = +\infty}$ $\quad \liminf_{n\to\infty} s_n = +\infty \Rightarrow \forall M>0\ \exists N\ \forall n > N\ \ u_n > M$

Then $u_{N+1} = \inf\{s_n : n > N+1\} > M \Leftrightarrow \forall n > N+1\ \ s_n > M$.

$\boxed{s \in \mathbb{R}}$ $\quad \liminf_{n\to\infty} s_n = \lim_{n\to\infty} u_n = c$, $\limsup_{n\to\infty} s_n = \lim_{n\to\infty} v_n = c$

$$\forall n \geq 2 \quad u_{n-1} = \inf\{s_k : k > n-1\} \leq s_n \leq \sup\{s_k : k > n-1\} = v_{n-1}$$

$$u_{n-1} \xrightarrow{n\to\infty} s, \qquad s_n \xrightarrow{n\to\infty} s \text{ (Squeeze Lemma)}, \qquad v_{n-1} \xrightarrow{n\to\infty} s$$

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