MATH 142A: Introduction to Analysis

www.math.ucsd.edu/~ynemish/teaching/142a

Today: Monotone sequences

> Q&A: January 22

Next: Ross § 10

Week 3:

- Homework 2 (due Friday, January 22)
- Midterm 1 on Wednesday, January 27 (lectures 1-7)
- Regrades for HW1: Mon, Jan 25 - Tue, Jan 26 (PST) on Gradescope

Monotone sequences

Def 10.1 A sequence (s_n) is called

an increasing sequence if

a decreasing sequence if

a monotone / monotonic sequence if it is increasing or decreasing

Examples

$a_n = 0$

$b_n = n$

$c_n = -n$

$d_n = \frac{1}{n}$

$e_n = \frac{(-1)^n}{n}$

$f_n = \frac{n}{1+n}$

$g_n = n^2 - 4n$

$h_n = \left(1 + \frac{1}{n}\right)^n$

Bounded monotone sequences converge

Thm 10.2 All bounded monotone sequences converge.

Proof Let (s_n) be a bounded increasing sequence.

Denote $S := \{s_n : n \in \mathbb{N}\}$. Then

(s_n) bounded $\Rightarrow$

Fix $\varepsilon > 0$. Then

1

2

Important example : the number e

Sequence $a_n = \left(1+\frac{1}{n}\right)^n$ is convergent.

Proof. ① Sequence

② Sequence b_n is bounded below :

③ ① + ② + Thm. 10.2 $\Rightarrow$ sequence $(b_n)_{n=1}^{\infty}$

④ $\lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^n =$

Example

Consider the sequence $(a_n)_{n=1}^{\infty}$ given by $a_1 = \sqrt{5}$, $a_{n+1} = \sqrt{5 + a_n}$

Is (a_n) convergent? If yes, what is the limit?

① (a_n) is monotone

② (a_n) is bounded above

① + ② + Thm. 10.2 $\Rightarrow$

Unbounded monotone sequences

Thm 10.4 (i) If (s_n) is unbounded and increasing, then

(ii) If (s_n) is unbounded and decreasing, then

Proof (i) Fix $M > 0$. (s_n) unbounded $\Rightarrow$

(s_n) increasing $\Rightarrow$

Corollary 10.5 If (s_n) is a monotone sequence, then it has a limit

i.e., (s_n) converges or diverges to $+\infty$ or diverges to $-\infty$.

limsup and liminf

Let (s_n) be convergent, $\lim_{n\to\infty} s_n = s$. Then $\forall \varepsilon>0\ \exists N$

$\forall n>N\ |s_n - s| < \varepsilon$

$\lim_{n\to\infty} s_n = s$ iff $\forall \varepsilon>0\ \exists N$

$(u_n)_{n=1}^{\infty}$ is

$(v_n)_{n=1}^{\infty}$ is

$\Rightarrow \left(\lim_{n\to\infty} s_n = s \Rightarrow \right)$

Def 10.6 Let (s_n) be a sequence. We define

$$\limsup_{n\to\infty} = \overline{\lim_{n\to\infty}} :=$$

$$\liminf_{n\to\infty} = \underline{\lim_{n\to\infty}} :=$$

If $\sup\{s_n : n\in\mathbb{N}\} = +\infty$, $\limsup_{n\to\infty} s_n =$; if $\inf\{s_n : n\in\mathbb{N}\} = -\infty$, $\liminf_{n\to\infty} s_n =$

limsup and liminf

Examples 1) $a_n = n$, $\forall N$ $\sup\{a_n : n > N\}$ $\Rightarrow \limsup\limits_{n\to\infty} n$

$\forall N$ $\inf\{a_n : n > N\}$ $\Rightarrow \liminf\limits_{n\to\infty} n$

2) $b_n = \frac{1}{n}$, $\forall N$ $\sup\{b_n : n > N\}$ $\Rightarrow \limsup\limits_{n\to\infty} \frac{1}{n}$

$\forall N$ $\inf\{b_n : n > N\}$ $\Rightarrow \liminf\limits_{n\to\infty} \frac{1}{n}$

3) $c_n = \frac{(-1)^n}{n}$ $\forall N$ $\sup\{c_n : n > N\}$ $\Rightarrow \limsup\limits_{n\to\infty} \frac{(-1)^n}{n}$

$\forall N$ $\inf\{c_n : n > N\}$ $\Rightarrow \liminf\limits_{n\to\infty} \frac{(-1)^n}{n}$

4) $d_n = (-1)^n$ $\forall N$ $\sup\{d_n : n > N\}$ $\Rightarrow \limsup\limits_{n\to\infty} (-1)^n$

$\forall N$ $\inf\{d_n : n > N\}$ $\Rightarrow \liminf\limits_{n\to\infty} (-1)^n$

5) $e_n = n^{(-1)^n}$ $\forall N$ $\sup\{e_n : n > N\}$ $\Rightarrow \limsup\limits_{n\to\infty} n^{(-1)^n}$

$\forall N$ $\inf\{e_n : n > N\}$ $\Rightarrow \liminf\limits_{n\to\infty} n^{(-1)^n}$

Convergence and limsup/liminf

Thm. 10.7 Let (s_n) be a sequence in $\mathbb{R}$, $s \in \mathbb{R}$ or $s \in \{+\infty, -\infty\}$.

Then (i) $\lim_{n\to\infty} s_n = s \implies$

(ii) $\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = s \implies$

Proof Denote $u_N = \inf\{s_n : n > N\}$, $v_N = \sup\{s_n : n > N\}$, $u = \liminf_{n\to\infty} s_n$, $v = \limsup_{n\to\infty} s_n$

(i) Three cases: $s = +\infty$, $s = -\infty$, $s \in \mathbb{R}$

$\boxed{s = +\infty}$ Fix $M > 0$. Then $\exists N \ \forall n > N$, and this implies

that

On the other hand, $\lim_{n\to\infty} s_n = +\infty \implies$

$\boxed{s \in \mathbb{R}}$ Fix $\varepsilon > 0$. Then $\exists N \ \forall n > N$ $s - \varepsilon < s_n < s + \varepsilon$. Then

(a)

(b)

(c)

Convergence and limsup/liminf

(ii) Three cases: $s = +\infty$, $s = -\infty$, $s \in \mathbb{R}$.

$\boxed{s = +\infty}$ $\quad \liminf_{n\to\infty} s_n = +\infty \Rightarrow \forall M \ \exists N \ \forall n > N \quad u_n > M$

Then

$\boxed{s \in \mathbb{R}}$ $\quad \liminf_{n\to\infty} s_n = \lim_{n\to\infty} u_n = c, \quad \limsup_{n\to\infty} s_n = \lim_{n\to\infty} v_n = c$