

MATH 142A: Introduction to Analysis

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Today: Cauchy sequences

> Q&A: January 29

Next: Ross § 11

Week 4:

- Homework 3 (due Sunday, January 31)
- Midterm 1 on Wednesday, January 27 (lectures 1-7)

Cauchy sequences

Def 7.1. A sequence (s_n) of real numbers is said to **converge** to the real number s if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n > N (|s_n - s| < \varepsilon)$$

Def. 10.8 A sequence (s_n) is called a **Cauchy sequence** if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall m, n > N (|s_m - s_n| < \varepsilon)$$

Examples Fix $\varepsilon > 0$.

$$N > \lceil \frac{1}{\varepsilon} \rceil$$

1. $a_n = \frac{1}{n}$: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ $m, n > N \Rightarrow |a_m - a_n| = \left| \frac{1}{m} - \frac{1}{n} \right| < \frac{1}{N} < \varepsilon$

2. $b_n = \frac{(-1)^n}{n}$: $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \dots$ $m, n > N \Rightarrow |b_m - b_n| \leq |b_m| + |b_n| < \frac{1}{N} + \frac{1}{N} < \varepsilon$

3. $c_n = 1 + \frac{1}{n}$: $1, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \dots$ $m, n > N \Rightarrow |c_m - c_n| = \left| \frac{1}{m} - \frac{1}{n} \right| < \frac{1}{N} < \varepsilon$

Cauchy sequences

Lemma 10.9 Convergent sequences are Cauchy sequences.

Proof. Suppose $\lim_{n \rightarrow \infty} s_n = s \in \mathbb{R}$. Fix $\varepsilon > 0$.

Then $\exists N \forall n > N \left(|s_n - s| < \frac{\varepsilon}{2} \right)$

Using the triangle inequality, $\forall m, n \in \mathbb{N} \left(|s_n - s_m| \leq |s_n - s| + |s - s_m| \right)$

Therefore, $\forall m, n > N \left(|s_n - s_m| \leq |s_n - s| + |s_m - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \right)$

Lemma 10.10 Cauchy sequences are bounded.

Proof. Suppose (s_n) is a Cauchy sequence. Then (take $\varepsilon = 1$)

$\exists N \in \mathbb{N} \forall m, n > N \left(|s_n - s_m| < 1 \right)$. Now take $m = N+1$

$\forall n > N \left(|s_n - s_{N+1}| < 1 \Rightarrow \forall n > N \left(|s_n| \leq |s_n - s_{N+1}| + |s_{N+1}| < |s_{N+1}| + 1 \right) \right)$

With $M := \max \{ |s_1|, |s_2|, \dots, |s_N|, |s_{N+1}| + 1 \}$ we have $\forall n \left(|s_n| \leq M \right)$.

Cauchy sequences converge

Thm 10.11 (s_n) converges $\Leftrightarrow (s_n)$ is a Cauchy sequence

Proof (\Rightarrow) Lemma 10.9.

(\Leftarrow) Suppose (s_n) is a Cauchy sequence.

By Lemma 10.10 (s_n) is bounded. Therefore, by Thm 10.7

it is enough to show that $\liminf_{n \rightarrow \infty} s_n = \limsup_{n \rightarrow \infty} s_n$

Denote $u_n = \inf \{s_k : k > n\}$, $v_n = \sup \{s_k : k > n\}$.

Fix $\varepsilon > 0$. Then $\exists N \forall m, n > N (|s_m - s_n| < \frac{\varepsilon}{3})$. In particular

$$\forall m, n > N (s_n < s_m + \frac{\varepsilon}{3}) \Rightarrow \forall m > N \quad v_N \leq s_m + \frac{\varepsilon}{3}$$

$$\text{Similarly, } \forall m, n > N (s_n - \frac{\varepsilon}{3} < s_m) \Rightarrow \forall n > N \quad u_N \geq s_n - \frac{\varepsilon}{3}$$

Take $k > N$. Then $|v_k - u_k| = v_k - u_k \leq v_N - u_N \leq s_m + \frac{\varepsilon}{3} - s_n + \frac{\varepsilon}{3} < \varepsilon$

Therefore, $\lim_{k \rightarrow \infty} (v_k - u_k) = 0 = \lim_{k \rightarrow \infty} v_k - \lim_{k \rightarrow \infty} u_k = \limsup_{n \rightarrow \infty} s_n - \liminf_{n \rightarrow \infty} s_n$ ■

Examples

1) Let $a_n = \frac{\cos(1)}{2} + \frac{\cos(2)}{2^2} + \frac{\cos(3)}{2^3} + \dots + \frac{\cos(n)}{2^n}$. Then

(a_n) is a Cauchy sequence and thus (a_n) converges.

Proof. Fix $\varepsilon > 0$. Then $\forall m > n > N$

$$|a_m - a_n| = \left| \frac{\cos(1)}{2} + \frac{\cos(2)}{2^2} + \dots + \frac{\cos(n)}{2^n} + \dots + \frac{\cos(m)}{2^m} - \left(\frac{\cos(1)}{2} + \dots + \frac{\cos(n)}{2^n} \right) \right|$$

$$= \left| \frac{\cos(n+1)}{2^{n+1}} + \dots + \frac{\cos(m)}{2^m} \right| \leq \left| \frac{\cos(n+1)}{2^{n+1}} \right| + \dots + \left| \frac{\cos(m)}{2^m} \right|$$

$$\leq \frac{1}{2^{n+1}} + \dots + \frac{1}{2^m} < \frac{1}{2^n} \quad [< \varepsilon]$$

$\exists N \forall n > N \left(\frac{1}{2^n} < \varepsilon \right) \Rightarrow \forall m, n > N \quad |a_m - a_n| < \frac{1}{2^n} < \varepsilon \Rightarrow$ (a_n) is a Cauchy sequence. \blacksquare

2) Let $b_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$. Take $\varepsilon = \frac{1}{2}$. Then $\forall n$

$$|b_{2n} - b_n| = \underbrace{1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}}_n + \frac{1}{n+1} + \dots + \frac{1}{2n} - \left(1 + \frac{1}{2} + \dots + \frac{1}{n} \right) = \frac{1}{n+1} + \dots + \frac{1}{2n}$$

$$> \frac{1}{2n} + \frac{1}{2n} + \dots + \frac{1}{2n} = \frac{n}{2n} = \frac{1}{2}$$

$\Rightarrow \forall N \quad |b_{2(N+1)} - b_{N+1}| > \frac{1}{2} \Rightarrow (b_n)$ is not a Cauchy sequence \Rightarrow does not converge. \blacksquare

Asymptotic behavior of sequences

Lemma 10.12 (Exercise 9.12) (s_n) $(-1)^n$

Assume that all $s_n \neq 0$ and that $\lim_{n \rightarrow \infty} \left| \frac{s_{n+1}}{s_n} \right| = L \in [0, +\infty)$.

(a) If $L < 1$, then $\lim_{n \rightarrow \infty} s_n = 0$

(b) if $L > 1$, then $\lim_{n \rightarrow \infty} |s_n| = +\infty$ (Use part (a) and Thms 9.5 & 9.10)

Proof. Let $L \in [0, 1)$. Fix $\varepsilon > 0$. Take $a \in (L, 1)$, $L < a < 1$.

Then by Thm 9.11 (i) (Lec 6) $\exists N \forall n > N \left(\left| \frac{s_{n+1}}{s_n} \right| < a \right)$

In particular, $|s_{n+2}| < a |s_{n+1}|$, $|s_{n+3}| < a |s_{n+2}| < a^2 |s_{n+1}|$, ..., $|s_{n+k}| < a^{k-1} |s_{n+1}|$

Consider the sequence (b_n) with $b_n = a^n \frac{|s_{n+1}|}{a^{n+1}}$. Then

(i) by Thm 9.2 (Lec 5) and Important example 2 (Lec 6), $\lim_{n \rightarrow \infty} b_n = 0$

(ii) $\forall n$ $0 < |s_n| \leq b_n$, therefore by Thm 9.11 (ii) Lec 6 $\lim_{n \rightarrow \infty} |s_n| = 0$

Finally, $\forall n$ $-|s_n| \leq s_n \leq |s_n| \Rightarrow \lim_{n \rightarrow \infty} s_n = 0$.

Example

Exercise 9.13.

$$\lim_{n \rightarrow \infty} a^n = \begin{cases} 0, & |a| < 1 \\ 1, & a = 1 \\ +\infty, & a > 1 \\ \text{DNE}, & a \leq -1 \end{cases}$$

Proof. Case $|a| < 1$: Consider the sequence $\left(\left| \frac{a^{n+1}}{a^n} \right| \right)_{n=1}^{\infty}$.

$$\forall n \in \mathbb{N} \quad \left| \frac{a^{n+1}}{a^n} \right| = |a| \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{a^{n+1}}{a^n} \right| = |a| < 1 \stackrel{L10.12}{\Rightarrow} \lim_{n \rightarrow \infty} a^n = 0$$

Case $a = 1$: $\forall n \quad a^n = 1 \Rightarrow \lim_{n \rightarrow \infty} a^n = \lim_{n \rightarrow \infty} 1 = 1$

Case $a > 1$: $\lim_{n \rightarrow \infty} \left| \frac{a^{n+1}}{a^n} \right| = \lim_{n \rightarrow \infty} a = a > 1 \stackrel{L10.12}{\Rightarrow} \lim_{n \rightarrow \infty} |a^n| = \lim_{n \rightarrow \infty} a^n = +\infty$.

Case $a \leq -1$: Denote $b = -a \geq 1$, so that $a^n = (-1)^n \cdot b^n$, Note $\forall n \quad b^n \geq 1$

Then $\forall N \in \mathbb{N} \cdot \exists n_1 > N$ (enough to take $n_1 = 2k > N$) $a^{n_1} = (-1)^{2k} b^{2k} \geq 1$
 $\cdot \exists n_2 > N$ (enough to take $n_2 = 2k+1 > N$) $a^{n_2} = (-1)^{2k+1} b^{2k+1} \leq -1$

Therefore, $\limsup_{n \rightarrow \infty} a^n = 1 \neq -1 = \liminf_{n \rightarrow \infty} a^n \Rightarrow (a^n)$ is divergent.

Important example 6 (asymptotic growth).

For any $p \in \mathbb{N}$ and any $a > 1$

$$\lim_{n \rightarrow \infty} \frac{n^p}{a^n} = 0$$

(exponential sequences grow to ∞ faster than polynomial sequences)

Proof. Denote $x_n := \frac{n^p}{a^n}$ Then

$$\frac{x_{n+1}}{x_n} = \frac{(n+1)^p}{a^{n+1}} \cdot \frac{a^n}{n^p} = \frac{1}{a} \left(\frac{n+1}{n} \right)^p = \frac{1}{a} \cdot \left(1 + \frac{1}{n} \right)^p$$

① $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) = 1$

② By Thm 9.4 + ① (applied $p-1$ times) $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^p = 1$

$$\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{x_{n+1}}{x_n} \right| = \lim_{n \rightarrow \infty} \frac{1}{a} \cdot \left(1 + \frac{1}{n} \right)^p = \frac{1}{a} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^p = \frac{1}{a} < 1$$

③ By Lemma 10.12, $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \frac{n^p}{a^n} = 0$ ■.

Important example 7 (asymptotic growth)

For any $a > 1$

$$\lim_{n \rightarrow \infty} \frac{a^n}{n!} = 0$$

(factorial grows to ∞ faster than any exponential sequence)

Proof. Denote $y_n = \frac{a^n}{n!}$

$$\textcircled{1} \quad \lim_{n \rightarrow \infty} \left| \frac{y_{n+1}}{y_n} \right| = \lim_{n \rightarrow \infty} \frac{a^{n+1}}{(n+1)!} \cdot \frac{n!}{a^n} = \lim_{n \rightarrow \infty} \frac{a}{n+1} = 0$$

$$\textcircled{2} \quad \text{By Lemma 10.12, } \lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} \frac{a^n}{n!} = 0.$$

