MATH 142A: Introduction to Analysis

www.math.ucsd.edu/~ynemish/teaching/142a

Today: Subsequences

> Q&A: Jan 29, Feb 1

Next: Ross § 11-12

Week 4:

- Homework 3 (due Sunday, January 31)

Subsequences

$a_n = (-1)^n, n \geq 1 : -1, 1, -1, 1, -1, 1, -1, 1, \ldots$

$b_n = \cos(\frac{\pi n}{2}), n \geq 1 : 0, -1, 0, 1, 0, -1, 0, 1, 0, -1, 0, 1, \ldots)$

$c_n = n, n \geq 1 : 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, \ldots$

$d_n = \cos(n), n \geq 1 : \cos(1), \cos(2), \cos(3), \cos(4), \cos(5), \cos(6), \ldots$

Def 11.1 Let (s_n) be a sequence of real numbers and let

be an increasing sequence of natural numbers.

Then

is called a

Subsequences

Thm 11.2 Let (s_n) be a sequence. Let $t \in \mathbb{R}$.

(i) There exists a (monotonic) subsequence of (s_n) converging to t

$$\Leftrightarrow$$

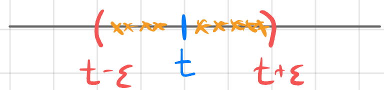
Proof $(\Rightarrow)$ Exercise.

$(\Leftarrow)$ $\forall \varepsilon > 0$ the set $\{n \in \mathbb{N} : |s_n - t| < \varepsilon\}$ is infinite.

Case 1: the set $\{n : s_n = t\}$ is infinite, take (s_{n_k}) with $s_{n_k} = t$ $\forall k$.

Case 2: $\forall \varepsilon > 0$ the set is infinite.

Either (a) $\forall \varepsilon > 0$ is infinite



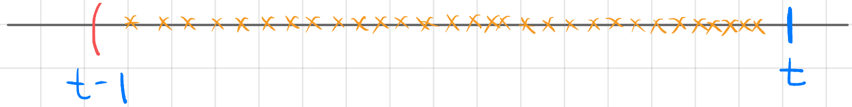
or (b) $\forall \varepsilon > 0$ is infinite

Consider Case 2(a). We want to construct an increasing subsequence that converges to t.

Proof of Thm 11.2 (i)

Suppose that $\forall \varepsilon > 0$ $\{n : t-\varepsilon < s_n < t\}$ is infinite

① Choose n_1 such that



② Take , so that

, and thus the set

is infinite.

Then

Choose

ⓚ Suppose we have numbers $n_1 < n_2 < \cdots < n_{k-1}$ such that

Take

$\{n : t-\varepsilon < s_n < t\}$ is infinite $\Rightarrow$

$(s_{n_k})_{k=1}^{\infty}$ is a subsequence of $(s_n)_{n=1}^{\infty}$, and

Subsequences

Thm 11.2 Let (s_n) be a sequence.

(ii) (s_n) has a (monotonic) subsequence that diverges to $+\infty$

$\Longleftrightarrow$

(iii) (s_n) has a (monotonic) subsequence that diverges to $-\infty$

$\Longleftrightarrow$

Proof (ii) $(\Rightarrow)$ Exercise.

$(\Leftarrow)$ Suppose that (s_n) is unbounded above.

① Let , so that

② (s_n) unbounded above $\Rightarrow$ is infinite

choose

⋮

Ⓚ is infinite, choose

Then (s_{n_k}) is a subsequence, $\forall k$

Subsequences

Thm 11.3 If (s_n) converges, then any subsequence of (s_n) converges to the same limit.

Proof. Let (s_{n_k}) be a subsequence of (s_n).

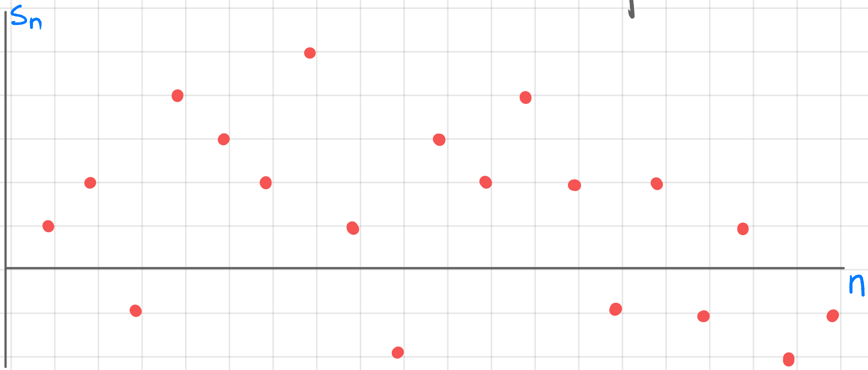
①

Proof by induction:

② Suppose (s_n) converges to $s \in \mathbb{R}$. Fix $\varepsilon > 0$. Then

Subsequences

Thm 11.4 Every sequence has a monotonic subsequence.



Proof Let (s_n) be a sequence of real numbers.

We say that s_n is

if

Denote $D =$

Case 1: D is infinite. Take

Then $n_1 < n_2$, $n_{k-1} < n_k$

Case 2: D is finite. Take Then

Bolzano-Weierstrass Theorem

Thm. 11.5 Every bounded sequence has a convergent subsequence.

Proof Let (s_n) be a bounded sequence.

By Thm 11.4

Since (s_n) is bounded,

(s_{n_k}) is monotonic and bounded, therefore