MATH 142A - INTRODUCTION TO ANALYSIS
PRACTICE MIDTERM 1

WINTER 2021

Name (Last, First): ______________________________

Student ID: ______________________________

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT.

1. Prove that for any $n \in \mathbb{N}$

$$(2n)! < 2^{2n}(n!)^2. \tag{1}$$

Solution. We prove this statement using the principle of mathematical induction.
Basis of induction, $n = 1$:

$$(2 \cdot 1)! = 2! = 2 < 2^{2 \cdot 1}(1!)^2 = 4. \tag{2}$$

Induction step. Suppose that

$$(2n)! < 2^{2n}(n!)^2. \tag{3}$$

Note that for any $n \in \mathbb{N}$

$$(2n+1)(2n+2) < (2(n+1))^2. \tag{4}$$

Therefore,

$$(2(n+1))! = (2n)!(2n+1)(2n+2) < 2^{2n}(n!)^2(2n+2)^2 = 2^{2(n+1)}((n+1)!)^2. \tag{5}$$

By the principle of mathematical induction, for any $n \in \mathbb{N}$

$$(2n)! < 2^{2n}(n!)^2. \tag{6}$$

2. Prove that the set

$$S := \left\{ \frac{n}{n+3}(2 + (-1)^n) \, : \, n \in \mathbb{N} \right\} \tag{7}$$

is bounded. Determine $\sup S$ and $\inf S$ (provide the proof).

Solution.
Step 1: S is bounded. First, note that for any $n \in \mathbb{N}$

$$1 = 2 - 1 \leq 2 + (-1)^n \leq 2 + 1 = 3. \tag{8}$$

Note that for any $n \in \mathbb{N}$

$$n + 3 \leq 4n, \tag{9}$$

which implies that for any $n \in \mathbb{N}$

$$\frac{n}{n+3} \geq \frac{1}{4}. \tag{10}$$

Moreover, for any $n \in \mathbb{N}$

$$\frac{n}{n+3} < \frac{n}{n} = 1 \tag{11}$$

We conclude that for any $n \in \mathbb{N}$

$$\frac{1}{4} \leq \frac{n}{n+3} \cdot 1 \leq \frac{n}{n+3}(2 + (-1)^n) < \frac{n}{n} \cdot 3 = 3, \tag{12}$$

and thus for any $x \in S$

$$\frac{1}{4} \leq x < 3. \tag{13}$$

Step 2: $\sup S = 3$. Suppose that $M < 3$. By the Archimedean property there exists $n_0 \in \mathbb{N}$ such that

$$2n_0 > \frac{3M}{3-M} > 0. \tag{14}$$

Note that

$$2n_0 > \frac{3M}{3-M} \quad \Leftrightarrow \quad \frac{3 \cdot 2n_0}{2n_0+3} > M. \tag{15}$$

Therefore, if $M < 3$ there exists an element $x \in S$

$$x = \frac{2n_0}{2n_0+3}(2+(-1)^{2n_0}) \tag{16}$$

such that $x > M$. We conclude that $\sup S = 3$.

Step 3: $\inf S = \frac{1}{4}$. Note that for $n = 1$

$$\frac{1}{1+3}(1+(-1)^1) = \frac{1}{4} \in S, \tag{17}$$

and as was shown in *Step 1*, for all $x \in S$, $\frac{1}{4} \leq x$. We conclude that $\frac{1}{4} = \min S$. By Example 3 (a) from the textbook, $\inf S = \min S = \frac{1}{4}$.

3. By checking the definition of a convergent sequence, compute the limit of the sequence $(a_n)_{n=1}^{\infty}$ with

$$a_n = \sqrt{n+1} - \sqrt{n}. \tag{18}$$

Solution. Firstly, note that

$$a_n = \sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}. \tag{19}$$

Now check the definition that $\lim_{n\to\infty} a_n = 0$. Fix $\varepsilon > 0$. Then for any $n > \left[\frac{1}{\varepsilon^2}\right]$

$$|a_n - 0| = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{\sqrt{n}} < \varepsilon. \tag{20}$$

We conclude, that for any $n > N(\varepsilon) := \left[\frac{1}{\varepsilon^2}\right]$, $|a_n - 0| < \varepsilon$, and thus $\lim_{n\to\infty} a_n = 0$.

4. Determine

$$\lim_{n\to\infty} \left(\frac{1}{2n} - \frac{2}{2n} + \frac{3}{2n} - \cdots + \frac{2n-1}{2n} - \frac{2n}{2n}\right). \tag{21}$$

Solution. Note that for all $n \in \mathbb{N}$

$$\left(\frac{1}{2n} - \frac{2}{2n} + \frac{3}{2n} - \cdots + \frac{2n-1}{2n} - \frac{2n}{2n}\right) \tag{22}$$

$$= \frac{1-2+3-4+\cdots+2n-1-2n}{2n} = \frac{-n}{2n} = -\frac{1}{2}. \tag{23}$$

Therefore,

$$\lim_{n\to\infty} \left(\frac{1}{2n} - \frac{2}{2n} + \frac{3}{2n} - \cdots + \frac{2n-1}{2n} - \frac{2n}{2n}\right) = -\frac{1}{2}. \tag{24}$$

5. Prove that the sequence $(b_n)_{n=1}^{\infty}$ with

$$b_n = 1 + \frac{1}{2\cdot 2} + \frac{1}{3\cdot 2^2} + \frac{1}{4\cdot 2^3} + \cdots + \frac{1}{n\cdot 2^{n-1}} \tag{25}$$

is convergent.

Solution. *Step 1: sequence (b_n) is increasing.* For any $n \in \mathbb{N}$

$$b_{n+1} - b_n = \frac{1}{(n+1)2^n} > 0, \tag{26}$$

therefore, $b_{n+1} > b_n$, the sequence is increasing.

Step 2: sequence (b_n) is bounded. Note that for any $n \in \mathbb{N}$

$$\frac{1}{n2^{n-1}} < \frac{1}{2^{n-1}}. \tag{27}$$

Using the above inequality, we have that for any $n \in \mathbb{N}$

$$b_n < 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^{n-1} = \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} < \frac{1}{\frac{1}{2}} = 2. \tag{28}$$

Therefore, sequence (b_n) is bounded above.

Step 3. Theorem 10.2 implies that sequence (b_n) is convergent.