MATH 142A - INTRODUCTION TO ANALYSIS
PRACTICE MIDTERM 2

WINTER 2021

Name (Last, First): ______________________________

Student ID: ______________________________

1. Let (s_n) be a monotonic sequence and let (s_{n_k}) be its subsequence. Prove that if the subsequence (s_{n_k}) is a Cauchy sequence, then (s_n) converges.

Solution.
Step 1: (s_{n_k}) *is bounded.* The sequence (s_{n_k}) is a Cauchy sequence. By Lemma 10.10 (s_{n_k}) is bounded This means that there exists a number $M > 0$ such that $|s_{n_k}| \le M$ for all $k \in \mathbb{N}$.
Step 2: (s_n) *is bounded.* By the definition of a subsequence, $k \le n_k$ for any $k \in \mathbb{N}$. If (s_n) is increasing, then for all $k \in \mathbb{N}$

$$k \le n_k \Rightarrow s_k \le s_{n_k} \le M, \tag{1}$$

and thus (s_n) is bounded above. If (s_n) is decreasing, then for all $k \in \mathbb{N}$

$$k \le n_k \Rightarrow s_k \ge s_{n_k} \ge -M, \tag{2}$$

and thus (s_n) is bounded below.
Step 3: (s_n) *converges.* Sequence (s_n) is monotonic and bounded, therefore by Theorem 10.2 (s_n) converges.

2. Determine the set of the partial limits, $\liminf$ and $\limsup$ of the sequence (x_n) given by

$$x_n = \frac{(-1)^n}{n} + \frac{1 + (-1)^n}{2}. \tag{3}$$

Remark. Partial limit is another term used to describe the subsequential limit.

Solution. Denote

$$s_n = \frac{(-1)^n}{n}, \quad t_n = \frac{1 + (-1)^n}{2}, \tag{4}$$

so that $x_n = s_n + t_n$. Denote by X and T the sets of the subsequential limits of the sequences (x_n) and (t_n) correspondingly.
Step 1: $X = T$*.* Sequence (s_n) converges to 0, therefore by Theorem 11.3, any subsequence of (s_n) converges to 0. If either the subsequence (x_{n_k}) or the subsequence (t_{n_k}) converges, then by Theorem 9.2

$$\lim x_{n_k} = \lim(t_{n_k} + s_{n_k}) = \lim(x_{n_k} - s_{n_k}) = \lim t_{n_k}, \tag{5}$$

and thus $X = T$.
Step 2: $T = \{0, 1\}$*.* We have that

$$t_{2n-1} = 0, \quad t_{2n} = 1, \tag{6}$$

therefore $\{0, 1\} \subset T$. If $t \notin \{0, 1\}$, then

$$\forall n \in \mathbb{N} \quad |t_n - t| \ge \min\{|t|, |1 - t|\} > 0, \tag{7}$$

so $t \notin T$.
We conclude that $X = T = \{0, 1\}$.
Step 3: $\liminf x_n = 0$*,* $\limsup x_n = 1$*.* Follows from Theorem 11.8 (ii).

3. Determine if the following series converge

(a)

$$\sum_{n=2}^{\infty} \frac{3}{\log n}$$

(b)

$$\sum_{n=2}^{\infty} \frac{3^n}{(\log n)^n}$$

Solution.

(a) By the Important Example 6

$$\lim \frac{n}{e^n} = 0, \tag{8}$$

therefore there exists $N \in \mathbb{N}$ such that for any $n > N$

$$n < e^n. \tag{9}$$

Function $x \mapsto \log x$ is increasing, so for any $n > N$

$$\log n < n \quad \left(\Leftrightarrow \quad \frac{1}{\log n} > \frac{1}{n} \right). \tag{10}$$

Since

$$\sum \frac{1}{n} = +\infty \tag{11}$$

by the comparison test (Theorem 14.6 (ii)) we have that

$$\sum \frac{3}{\log n} = +\infty. \tag{12}$$

(b) In order to establish the convergence of the series, use the root test (Theorem 14.9)

$$\lim \sqrt[n]{\frac{3^n}{\log^n n}} = \lim \frac{3}{\log n} = 0. \tag{13}$$

This implies that the series $\sum \frac{3^n}{\log^n n}$ converges.

4. Prove that the function

$$f(x) = 2^{\frac{1}{1+x^2}}$$

is continuous on $\mathbb{R}$.

Solution. *Step 1: Function $x \mapsto \frac{1}{1+x^2}$ is continuous on $\mathbb{R}$.* By Theorems 17.4, $g(x) = 1 + x^2$ is continuous on $\mathbb{R}$. Since $g(x) \geq 1$ for all $x \in \mathbb{R}$, by Theorem 17.4, $1/g$ is continuous of $\mathbb{R}$.

Step 2: Function $x \mapsto 2^x$ is continuous on $\mathbb{R}$. As stated in Lecture 16 (and proven in the Important Example 11).

Step 3: f is continuous on $\mathbb{R}$. Follows from Steps 1, 2 and Theorem 17.5 about the continuity of a composition of continuous functions.

5. Let $S \subset \mathbb{R}$ and let $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ be uniformly continuous on S. Prove that $f + g$ is uniformly continuous on S.

Solution. Fix $\varepsilon > 0$. From the definition of the uniform continuity, for any $\varepsilon > 0$ there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|x - y| < \delta_1 \quad \Rightarrow \quad |f(x) - f(y)| < \frac{\varepsilon}{2}, \tag{14}$$

$$|x - y| < \delta_2 \quad \Rightarrow \quad |g(x) - g(y)| < \frac{\varepsilon}{2}. \tag{15}$$

Take $\delta = \min\{\delta_1, \delta_2\}$. Then for all $x, y \in \mathbb{R}$ such that $|x - y| < \delta$ by using the triangle inequality we have

$$|f(x) + g(x) - (f(y) - g(y))| = |f(x) - f(y) + g(x) - g(y)| \tag{16}$$

$$\leq |f(x) - f(y)| + |g(x) - g(y)| \tag{17}$$

$$< \varepsilon. \tag{18}$$

This means that

$$|x - y| < \delta \quad \Rightarrow \quad |(f + g)(x) - (f + g)(y)| < \varepsilon, \tag{19}$$

function $f + g$ is uniformly continuous on $\mathbb{R}$.