☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem. Different parts of the same problem can be written on the same page (for example, part (a) and part (b))**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

☐ **From the moment you access the midterm problems on Gradescope you have 65 MINUTES to COMPLETE AND UPLOAD your exam to Gradescope. Plan your time accordingly.**

☐ **All steps of the proofs should be INCLUDED in your solutions. Provide references to the theorem/examples from the lectures/texbook used in your proofs.**

☐ **You are allowed to use the textbook, lecture notes and your personal notes. You are not allowed to use the electronic devices (except for accessing the online version of the textbook) or outside assistance. Outside assistance includes but is not limited to other people, the internet and unauthorized notes.**

1. (25 points) Let (a_n), (b_n) and (c_n) be three sequences of real numbers satisfying

$$a_n \le b_n \le c_n$$

for all $n \in \mathbb{N}$. Suppose that $\lim a_n = a$, $\lim c_n = c$, where a and c are two real numbers, $a < c$.

Let S denote the set of the *subsequential limits* of (b_n). Prove that $S \subset [a, c]$.

Solution. Denote by S the set of the subsequential limits of (b_n). Let (b_{n_k}) be a convergent subsequence of (b_n), $\lim_{k\to\infty} b_{n_k} = b \in S$. Consider the corresponding subsequences (a_{n_k}) and (c_{n_k}) of (a_n) and (c_n).

Since $\lim a_n = a$, by Theorem 11.3 any subsequence on (a_n) converges to a. In particular

$$\lim_{k\to\infty} a_{n_k} = a. \tag{1}$$

Similarly, by Theorem 11.3

$$\lim_{k\to\infty} c_{n_k} = c. \tag{2}$$

It is given that $a_n \le b_n \le c_n$ for any $n \in \mathbb{N}$, therefore for any $k \in \mathbb{N}$

$$a_{n_k} \le b_{n_k} \le c_{n_k}. \tag{3}$$

By the corollary to the squeeze lemma, (1), (2) and (3) imply that

$$a \le \lim_{k\to\infty} b_{n_k} \le c. \tag{4}$$

Therefore, if $b \in S$, then $b \in [a, c]$, and thus $S \subset [a, c]$.

2. (25 points) Let (x_n) be a sequence of real numbers given by

$$x_n = (-1)^n \left(1 + \frac{1}{n}\right)^n + \sin\frac{\pi n}{2}$$

for $n \in \mathbb{N}$. Determine the set of the subsequential limits of (x_n), $\limsup x_n$ and $\liminf x_n$.

Solution. Denote by S the set of the subsequential limits of (x_n), and denote

$$a_n = (-1)^n, \quad b_n = \left(1 + \frac{1}{n}\right)^n, \quad c_n = \sin\frac{\pi n}{2}. \tag{5}$$

By the important example from Lecture 7, $\lim b_n = e$, and by Theorem 11.3 each subsequence of (b_n) converges to e.

If $n_k = 2k$, then $a_{n_k} = 1$, $c_{n_k} = 0$ for all $k \in \mathbb{N}$, and $\lim_{k\to\infty} x_{n_k} = e$.

If $n_k = 4k - 3$, then $a_{n_k} = -1$, $c_{n_k} = 1$ for all $k \in \mathbb{N}$, and $\lim_{k\to\infty} x_{n_k} = -e + 1$.

If $n_k = 4k-1$, then $a_{n_k} = -1$, $c_{n_k} = -1$ for all $k \in \mathbb{N}$, and $\lim_{k\to\infty} x_{n_k} = -e-1$.
We conclude that $S \supset \{e, -e+1, -e-1\}$.
If $s \notin \{e, -e-1, -e+1\}$, denote $\delta := \min\{|e-s|, |-e-1-s|, |-e+1-s|\} > 0$.
Take $N > 0$, such that $|b_n - e| < \frac{\delta}{2}$ for all $n > N$.
If $n > N$ and $n = 2k$, then

$$|x_{2k} - s| = |b_{2k} - s| \geq |e-s| - |b_{2k} - e| > \delta - \frac{\delta}{2} = \frac{\delta}{2} > 0. \tag{6}$$

If $n > N$ and $n = 4k-3$, then

$$|x_{4k-3} - s| = |-b_{4k-3} + 1 - s| \geq |-e+1-s| - |b_{2k} - e| > \delta - \frac{\delta}{2} = \frac{\delta}{2} > 0. \tag{7}$$

If $n > N$ and $n = 4k-1$, then

$$|x_{4k-1} - s| = |-b_{4k-1} - 1 - s| \geq |-e-1-s| - |b_{4k-1} - e| > \delta - \frac{\delta}{2} = \frac{\delta}{2} > 0. \tag{8}$$

Therefore, $|x_n - s| > \frac{\delta}{2} > 0$ for all $n > N$, so $s \notin S$, and thus $S = \{e, -e+1, -e-1\}$.

3. (25 points) Determine if the series

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$$

converges.

Solution. Note that for all

$$\sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(\frac{n+1}{n}\right)^n} \tag{9}$$

therefore, by the important example from Lecture 7

$$\lim \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \lim \frac{1}{\left(\frac{n+1}{n}\right)^n} = \frac{1}{e} < 1. \tag{10}$$

By the root test the series converges.

4. (25 points) Prove that the function $f(x) = 7^x$ is not uniformly continuous on $\mathbb{R}$.

Solution. Consider the sequence (x_n) with

$$x_n = \log_7 n. \tag{11}$$

Then

$$x_{n+1} - x_n = \log_7(n+1) - \log_7 n = \log_7 \frac{n+1}{n}. \tag{12}$$

The function $x \mapsto \log_7 x$ is continuous on $\mathbb{R}$, therefore

$$\lim_{n\to\infty} \log_7 \frac{n+1}{n} = \log_7 \left(\lim_{n\to\infty} \frac{n+1}{n} \right) = \log_7 1 = 0, \tag{13}$$

therefore

$$\lim_{n\to\infty} (x_{n+1} - x_n) = 0. \tag{14}$$

On the other hand

$$f(x_{n+1}) - f(x_n) = 7^{\log_7(n+1)} - 7^{\log_7 n} = n + 1 - n = 1. \tag{15}$$

Therefore, for any $\delta > 0$, by (14) there exist $x_n, x_{n+1} \in \mathbb{R}$ such that $|x_{n+1} - x_n| < \delta$, but by (15) $|f(x_{n+1}) - f(x_n)| = 1$. This contradicts the definition of the uniform continuity of function f on $\mathbb{R}$.