MATH 142A - INTRODUCTION TO ANALYSIS
PRACTICE FINAL

WINTER 2021

1. Let $a, b, c \in \mathbb{R}$ be such that $a < b < c$ and $(c-a)(c-b) = (b-a)^2$. Show that

$$r := \frac{c-a}{b-a} \tag{1}$$

is not a rational number.

Hint: Show that r satisfies a polynomial equation with integer coefficients.

2. Using only Definition 9.8 prove that

$$\lim_{n\to\infty} \log_{10}(\log_{10} n) = +\infty. \tag{2}$$

Clearly indicate how you chose $N(M)$ for any $M > 0$, and write explicitly $N(2)$, $N(5)$, $N(10)$.

3. Determine if the series

$$\sum_{n=1}^{\infty} \frac{2^n n!}{n^n} \tag{3}$$

converges. Justify your answer.

4. Let $a \in \mathbb{R}$ and let $f : [a, +\infty) \to \mathbb{R}$ be a function such that

(i) $f \in C([a, +\infty))$

(ii) $\lim_{x\to+\infty} f(x) = p \in \mathbb{R}$

Prove that f is *uniformly continuous* on $[a, +\infty)$.

5. Compute the derivative of the function $f : (0, +\infty) \to \mathbb{R}$ given by

$$f(x) = x + x^x. \tag{4}$$

Provide all intermediate steps.

6. Prove that the inequality

$$py^{p-1}(x-y) \le x^p - y^p \le px^{p-1}(x-y) \tag{5}$$

holds for $0 < y < x$ and $p > 1$.

7. Let

$$f : \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big) \to \mathbb{R}, \quad f(x) = \log(\cos x). \tag{6}$$

Find a polynomial $P(x)$ such that

$$f(x) - P(x) = o(x^3) \quad \text{as} \quad x \to 0. \tag{7}$$