MATH 142A: Introduction to Analysis

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Today: Subsequential limits

> Q&A: January 31

Next: Ross § 14

Week 5:

- Homework 4 (due Sunday, February 6)
- Homework 2 regardes (Tuesday, February 1)

Subsequential limits

Def 11.1 Let (s_n) be a sequence of real numbers and let $1 \le n_1 < n_2 < \cdots < n_k < \cdots$ be an increasing sequence of natural numbers.

Then $(s_{n_k})_{k=1}^{\infty} = (s_{n_1}, s_{n_2}, s_{n_3}, \dots)$ is called a subsequence of $(s_n)_{n=1}^{\infty}$.

$$\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \frac{1}{13}, \frac{1}{14}, \dots\right)$$

$$\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \frac{1}{13}, \dots\right)$$

Def 11.6 Let (s_n) be a sequence in $\mathbb{R}$. A subsequential limit is any real number or symbol $+\infty$ or $-\infty$ that is the limit of some subsequence of (s_n)

Example • $a_n = (-1)^n$, $(-1, 1, -1, 1, \dots)$ $\quad \lim_{k\to\infty} a_{2k-1} = -1$, $\lim_{k\to\infty} a_{2k} = 1$

Example • $b_n = 2^{n(-1)^n}$, $\left(\frac{1}{2}, 2^2, \frac{1}{2^3}, 2^4, \dots\right)$ $\quad \lim_{k\to\infty} b_{2k-1} = \lim_{k\to\infty} \frac{1}{2^{2k-1}} = 0$

$$\lim_{k\to\infty} b_{2k} = \lim_{k\to\infty} 2^{2k} = +\infty$$

Subsequential limits and liminf / limsup

Thm 11.7 Let (s_n) be a sequence. Then there exist

(i) a monotonic subsequence of (s_n) that converges to $\limsup s_n$

(ii) a monotonic subsequence of (s_n) that converges to $\liminf s_n$

Proof. If (s_n) is not bounded above, then $\limsup s_n = +\infty$. And by Thm 11.2 (ii) there exist a subsequence of (s_n) that diverges to $+\infty$.

Suppose (s_n) is bounded above, $\limsup_{n\to\infty} s_n = t \in \mathbb{R}$

By Thm 11.2 (i) there exists a monotonic subsequence of (s_n) that converges to t iff $\forall \varepsilon > 0$ the set $\{n : |s_n - t| < \varepsilon\}$ is infinite. Fix $\varepsilon > 0$

$\limsup_{n\to\infty} s_n = t \Rightarrow \exists N \ \forall k > N \ \sup\{s_n : n > k\} < t + \varepsilon \Rightarrow \forall n > N+1 \ (s_n < t + \varepsilon)$

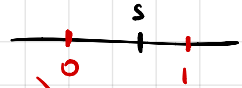
Suppose that $\{n : t - \varepsilon < s_n < t + \varepsilon\}$ is finite. Then $\exists N_1 > N$ s.t.

$\forall n > N_1 \ (s_n \le t - \varepsilon) \Rightarrow \sup\{s_n : n > N_1\} \le t - \varepsilon \Rightarrow \limsup_{n\to\infty} s_n \le t - \varepsilon$

contradiction

Subsequential limits and convergence

Thm 11.8. Let (s_n) be a sequence. Denote by S the set of all subsequential limits of (s_n). Then



(i) S is nonempty (follows Thm 11.7) $\quad (0,1,0,1,\dots)$ $\quad s\in\{0,1\}$ $\quad \forall n \ \ |s-s_n| \ge \min\{|s|,|s-1|\} > 0$

(ii) $\sup S = \limsup s_n$, $\quad \inf S = \liminf s_n$

(iii) $\lim s_n$ exists iff S has only one element, $S = \{\lim s_n\}$

Proof (iii) follows from (ii) and Thm 10.7 $\left(\lim s_n = t \iff \limsup s_n = \liminf s_n = t\right)$

(ii) Suppose $t \in S \iff \exists$ subsequence (s_{n_k}) s.t. $\lim_{k\to\infty} s_{n_k} = t$

Then by Thm 10.7 $\liminf_{k\to\infty} s_{n_k} = \limsup_{k\to\infty} s_{n_k} = t$

Note that $\forall k \ (n_k \ge k)$, therefore $\forall N \quad \{s_n : n>N\} \supset \{s_{n_k} : k>N\}$

and $\forall N \quad \inf\{s_n : n>N\} \le \inf\{s_{n_k} : k>N\} \le \sup\{s_{n_k} : k>N\} \le \sup\{s_n : n>N\}$

$N\to\infty$

$$\underset{\in S}{\liminf s_n} \overset{9.12}{\le} \liminf s_{n_k} = t = \limsup s_{n_k} \overset{9.12}{\le} \underset{\substack{\text{Thm 11.7} \\ \in S}}{\limsup s_n}$$

∎

Examples

For each sequence below let S denote the set of subsequential limits.

- $a_n = (-1)^n$,

① $S = \{-1, 1\}$

$\lim_{k \to +\infty} a_{2k-1} = -1$, $\lim_{k \to +\infty} a_{2k} = 1 \Rightarrow \{-1, 1\} \subset S$

If $t \notin \{-1, 1\}$, then $\forall n \in \mathbb{N}$ $|s_n - t| \geq \min\{|t-1|, |t-(-1)|\} > 0 \Rightarrow t \notin S$

$\Rightarrow S \subset \{-1, 1\}$

② $\limsup a_n = 1$ $\liminf a_n = -1$

- $b_n = 2^{n(-1)^n}$

① $S = \{0, +\infty\}$

$\lim_{k \to +\infty} b_{2k-1} = 0$, $\lim_{k \to +\infty} b_{2k} = +\infty \Rightarrow \{0, +\infty\} \subset S$

If $t \in \mathbb{R}$, $t \neq 0$, then $\exists N_1$ $\forall k > N_1$ $b_{2k-1} < \frac{t}{2}$

$\exists N_2$ $\forall k > N_2$ $b_{2k} > t + 1$

$\Rightarrow$ $\forall n > 2 \cdot \max\{N_1, N_2\}$ $|b_n - t| > \min\{\frac{t}{2}, 1\} > 0$

$S \subset \{0, +\infty\}$

② $\limsup b_n = +\infty$, $\liminf b_n = 0$

The set of subsequential limits is closed

Thm 11.9 Let (s_n) be a sequence. Denote by S the set of all subsequential limits of (s_n). Then

Let (t_n) be a sequence in $S \cap \mathbb{R}$, i.e. $\forall n \ (t_n \in S \cap \mathbb{R})$.
If (t_n) has a limit, then $\lim_{n\to\infty} t_n \in S$.

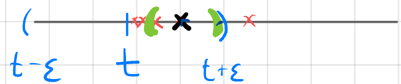
$x_n = \frac{1}{2n}$, $\forall n \ x_n \in (0,1)$, $\lim x_n = 0$

$0 \notin (0,1)$

Proof Suppose $\lim_{n\to\infty} t_n = t \in \mathbb{R}$. Then

$$t \in S \Leftrightarrow \exists \text{ subsequence of } (s_n) \text{ that converges to } t \overset{\text{Thm 11.2}}{\Longleftrightarrow} \forall \varepsilon > 0 \ \{n \in \mathbb{N} : |s_n - t| < \varepsilon\} \text{ infinite}$$

Fix $\varepsilon > 0$. Then $\exists n_0$ s.t. $t - \varepsilon < t_{n_0} < t + \varepsilon$



Since $t + \varepsilon - t_{n_0} > 0$, $t_{n_0} - (t - \varepsilon) > 0$

$$\delta := \min\{t + \varepsilon - t_{n_0}, \ t_{n_0} - (t - \varepsilon)\} > 0 \text{ and } (t_{n_0} - \delta, t_{n_0} + \delta) \subset (t - \varepsilon, t + \varepsilon)$$

$t_{n_0} \in S$ (subsequential limit) $\overset{\text{Thm 11.2}}{\Rightarrow} \{n : |s_n - t_{n_0}| < \delta\}$ is infinite $\Rightarrow \{n : |s_n - t| < \varepsilon\}$ is infinite $\blacksquare$

limsup's and liminf's

Thm 12.1 Let (s_n) and (t_n) be two sequences. Then

$$\left((s_n) \text{ converges} \wedge \lim_{n\to\infty} s_n = s > 0\right) \Rightarrow \limsup(s_n \cdot t_n) = s \cdot \limsup t_n$$

Convention: For any $s \in \mathbb{R}$, $s>0$, $s\cdot(+\infty) = +\infty$, $s\cdot(-\infty) = -\infty$.

Proof

① : $\limsup(s_n t_n) \geq s\cdot t$ (only for $\limsup t_n = t \in \mathbb{R}$)

Thm. 11.7 $\Rightarrow$ $\exists\,(t_{n_k})$ such that $\lim_{k\to\infty} t_{n_k} = t$. | Thm 9.4 $\Rightarrow$ $\lim_{k\to\infty}(s_{n_k} t_{n_k}) = s\cdot t$

Thm 11.3 $\Rightarrow$ $\lim_{k\to\infty} s_{n_k} = s$.

$\Rightarrow$ $s\cdot t$ is a subsequential limit for $(s_n t_n)$ $\overset{\text{Thm 11.8}}{\Longrightarrow}$ $\limsup(s_n t_n) \geq s\cdot t$

② : $\limsup(s_n t_n) \leq s\cdot t$ (only for $s_n > 0$ $\forall n$). Thm 9.5 $\Rightarrow$ $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s} > 0$

Then $t = \limsup t_n = \limsup \frac{1}{s_n}(s_n\cdot t_n) \overset{①}{\geq} \frac{1}{s} \limsup(s_n \cdot t_n) \Rightarrow$ $st \geq \limsup(s_n \cdot t_n)$

$\overset{①\,②}{\Longrightarrow}$ $\limsup(s_n t_n) = \limsup(t_n)\cdot s$ ■

Remark

If (s_n) and (t_n) are two sequences, and $\lim_{n\to\infty} s_n = 0$, then there is nothing we can say in general about $\limsup(s_n t_n)$.

- $s_n = \frac{1}{n}$, $t_n = n \Rightarrow \limsup \frac{1}{n}\cdot n = 1$
- $s_n = \frac{1}{n^2}$, $t_n = n \Rightarrow \limsup \frac{1}{n^2}\cdot n = 0$
- $s_n = \frac{1}{n}$, $t_n = n^2 \Rightarrow \limsup \frac{1}{n}\cdot n^2 = +\infty$

Also it is important that one sequence converges.

- $s_n = (0, 1, 0, 1, 0, 1, \dots)$ $\quad \limsup s_n = 1$
  $t_n = (1, 0, 1, 0, 1, 0, \dots)$ $\quad \limsup t_n = 1$
  but $\forall n \; s_n \cdot t_n = 0$
- $s_n = (-1)^n$, $t_n = (-1)^{n+1}$ $\quad \limsup s_n = \limsup t_n = 1$, but $\forall n \; s_n \cdot t_n = (-1)^{2n+1} = -1$