

MATH 142A: Introduction to Analysis

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Today: Series

> Q&A: February 2

Next: Ross § 15

Week 5:

- Homework 4 (due Sunday, February 6)

Sequences $\left| \frac{s_{n+1}}{s_n} \right|$ and $\sqrt[n]{|s_n|}$

Thm 12.2 Let (s_n) be a sequence, $\forall n (s_n \neq 0)$. Then

$$\liminf_{n \rightarrow \infty} \left| \frac{s_{n+1}}{s_n} \right| \leq \liminf_{n \rightarrow \infty} \sqrt[n]{|s_n|} \leq \limsup_{n \rightarrow \infty} \sqrt[n]{|s_n|} \leq \limsup_{n \rightarrow \infty} \left| \frac{s_{n+1}}{s_n} \right|$$

α β

Proof. If $l = 0$, then $l \leq \beta$. Assume that $l > 0$.

Take any $0 < l_1 < l$. Then by Thm 9.11 (i) $\exists N$ s.t.

$$\inf \left\{ \left| \frac{s_{n+1}}{s_n} \right| : n \geq N \right\} > l_1 \Rightarrow \forall n \geq N \left| \frac{s_{n+1}}{s_n} \right| > l_1.$$

$$\text{Therefore, } \forall n > N \quad |s_n| = |s_N| \cdot \frac{|s_{N+1}|}{|s_N|} \cdot \frac{|s_{N+2}|}{|s_{N+1}|} \cdots \frac{|s_n|}{|s_{n-1}|} > |s_N| \cdot l_1^{n-N} = \frac{|s_N|}{l_1^N} \cdot l_1^n$$

$$\Rightarrow \forall n > N \quad \sqrt[n]{|s_n|} > \sqrt[n]{\frac{|s_N|}{l_1^N} \cdot l_1^n} = l_1 \cdot \sqrt[n]{\frac{|s_N|}{l_1^N}} \Rightarrow \inf \left\{ \sqrt[n]{|s_n|} : n > N \right\} \geq l_1 \cdot \sqrt[n]{\frac{|s_N|}{l_1^N}} =: \tilde{u}_N$$

Note that (\tilde{u}_k) is increasing, so $\forall k > N \quad \tilde{u}_k \geq \tilde{u}_N \geq l_1 \cdot \sqrt[n]{\frac{|s_N|}{l_1^N}}$

$$\text{Now } \beta = \lim_{k \rightarrow \infty} \tilde{u}_k \stackrel{\text{Cor 9.12}}{\geq} \lim_{n \rightarrow \infty} \left(l_1 \cdot \sqrt[n]{\frac{|s_N|}{l_1^N}} \right) = l_1 \cdot \lim_{n \rightarrow \infty} \sqrt[n]{\frac{|s_N|}{l_1^N}} \stackrel{IE4}{=} l_1 \cdot 1 = l_1$$

So $\forall l_1 \in (0, l) \quad (\beta \geq l_1) \Rightarrow \beta$ is an upper bound for $(0, l) \Rightarrow \beta \geq l$.

Sequences $\left| \frac{s_{n+1}}{s_n} \right|$ and $\sqrt[n]{|s_n|}$

Corollary 12.3

If $\lim_{n \rightarrow \infty} \left| \frac{s_{n+1}}{s_n} \right|$ exists, and $\lim_{n \rightarrow \infty} \left| \frac{s_{n+1}}{s_n} \right| = L$, then $\lim_{n \rightarrow \infty} \sqrt[n]{|s_n|} = L$

Example

Let (a_n) be a sequence such that $\forall n \in \mathbb{N} \ a_n > 0$.

Suppose that (a_n) converges, $\lim_{n \rightarrow \infty} a_n = a$. Then

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_1 \cdot a_2 \cdots a_n} = a$$

Proof. Denote $s_n := a_1 \cdots a_n$. Then $\lim_{n \rightarrow \infty} \left| \frac{s_{n+1}}{s_n} \right| = \lim_{n \rightarrow \infty} a_{n+1} = a$

By Corollary 12.3 $\lim_{n \rightarrow \infty} \sqrt[n]{|s_n|} = a = \lim_{n \rightarrow \infty} \sqrt[n]{a_1 \cdots a_n}$

Series

Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers.

For $p, q \in \mathbb{N}$, $p < q$ we denote $a_p + a_{p+1} + \dots + a_q$ by $\sum_{n=p}^q a_n$

Def 14.1 (Infinite series) We call the expression

$$a_1 + a_2 + a_3 + \dots + a_n + \dots =: \sum_{n=1}^{\infty} a_n$$

an (infinite) series. a_n is called the n -th term of the series.

Def 14.2 (Convergent series)

We call the sum $S_n = \sum_{k=1}^n a_k$ the (n -th) partial sum of the series.

If the sequence (S_n) of partial sums converges, we say that the series $\sum_{n=1}^{\infty} a_n$ is convergent

If $\lim_{n \rightarrow \infty} S_n = s$, then we call s the sum of the series $\sum_{n=1}^{\infty} a_n$, and

write it as $\sum_{n=1}^{\infty} a_n = s$

Series

If $\lim_{n \rightarrow \infty} S_n = +\infty$ ($-\infty$), we say that $\sum_{n=1}^{\infty} a_n$ diverges to $+\infty$ ($-\infty$)
and we write $\sum_{n=1}^{\infty} a_n = +\infty$ (or $-\infty$)

We say that $\sum_{n=1}^{\infty} a_n$ converges absolutely (is absolutely convergent)

if the series $\sum_{n=1}^{\infty} |a_n|$ converges

Remark An infinite series can be viewed as a particular type of a sequence, $S_n = a_1 + a_2 + \dots + a_n$

so we can use all the relevant results.

For example, if $\forall n \ a_n \geq 0$, then S_n is increasing.

Partial sums of $\sum_{n=1}^{\infty} |a_n|$ form an increasing sequence.

Use the criteria on convergence for partial sums etc.

Important examples

8. Let $a, r \in \mathbb{R}$. Then $\sum_{n=0}^{\infty} ar^n$ is called the **geometric series**.

$$\text{If } |r| < 1, \text{ then } \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

Proof Denote $S_k = \sum_{n=0}^k ar^n = a + ar + \dots + ar^k = a(1 + r + r^2 + \dots + r^k)$

Note that $r(1 + r + \dots + r^k) = r + r^2 + \dots + r^k + r^{k+1}$, so

$$(1-r)(1+r+\dots+r^k) = 1+r+\dots+r^k - r(1+r+\dots+r^k) = 1-r^{k+1} \Rightarrow 1+r+\dots+r^k = \frac{1-r^{k+1}}{1-r}$$

$$\Rightarrow \forall k \quad S_k = \frac{a(1-r^{k+1})}{1-r}. \text{ By 1.E2 } \lim_{k \rightarrow \infty} r^{k+1} = 0 \Rightarrow \lim_{k \rightarrow \infty} S_k = \frac{a}{1-r} \quad \blacksquare$$

9. Let $p > 0$. Then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$

Proof ($p=2$). $S_k := \sum_{n=1}^k \frac{1}{n^2}$. ① (S_k) is increasing ($S_{k+1} - S_k = \frac{1}{(k+1)^2} > 0$)

② (S_k) is bounded $S_k = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{k^2} \leq 1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{(k-1)k}$

For any $n \geq 2$ $\frac{1}{(n-1)n} = \frac{1}{n-1} - \frac{1}{n}$, so $S_k \leq 1 + \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \dots + \frac{1}{k-1} - \frac{1}{k} = 2 - \frac{1}{k} < 2$

① + ② + Thm 10.2 \blacksquare

$\frac{6}{\pi^2}$

Cauchy criterion

Def 14.3 We say that $\sum_{n=1}^{\infty} a_n$ satisfies the Cauchy criterion if its sequence of partial sums (s_n) is a Cauchy sequence, i.e.

$$\forall \varepsilon > 0 \exists N \forall m, n > N |s_n - s_m| < \varepsilon$$

$$\forall \varepsilon > 0 \exists N \forall n > m > N \left| \sum_{k=m+1}^n a_k \right| < \varepsilon$$

Thm 14.4 $\sum a_n$ converges $\Leftrightarrow \sum a_n$ satisfies the Cauchy criterion

Proof. Follows from Thm 10.11

Corollary 14.5 (Necessary condition for convergence).

$$\sum a_n \text{ converges} \Rightarrow \lim_{n \rightarrow \infty} a_n = 0$$

Proof. $\sum a_n$ converges $\stackrel{\text{Thm 14.4}}{\Leftrightarrow} \forall \varepsilon > 0 \exists N \forall n > m > N \left| \sum_{k=m+1}^n a_k \right| < \varepsilon$ (take $n = m+1$)
 $\Rightarrow \forall \varepsilon > 0 \exists N \forall n > N+1 (|a_n| < \varepsilon) \Leftrightarrow \lim a_n = 0$ ■

Example

- $\sum_{k=1}^{\infty} \frac{1}{k 2^k}$ satisfies the Cauchy criterion

Proof. $\forall k \in \mathbb{N} \frac{1}{k \cdot 2^k} \leq \frac{1}{2^k}$, so $\forall n > m \geq 1$

$$\sum_{k=m+1}^n \frac{1}{k 2^k} \leq \sum_{k=m+1}^n \frac{1}{2^k} = \frac{1}{2^{m+1}} \sum_{l=0}^{n-m} \frac{1}{2^l} < \frac{1}{2^{m+1}} \sum_{l=0}^{\infty} \frac{1}{2^l} = \frac{1}{2^{m+1}} \cdot \frac{1}{1-\frac{1}{2}} = \frac{1}{2^m}$$

Fix $\varepsilon > 0$. By 1.E.2 $\exists N \forall m > N \left(\frac{1}{2^m} < \varepsilon \right)$ Therefore
 $\forall n > m > N \sum_{k=m+1}^n \frac{1}{k 2^k} \leq \frac{1}{2^m} < \varepsilon \Rightarrow \sum_{k=1}^{\infty} \frac{1}{k 2^k}$ satisfies the
Cauchy criterion

In particular, $\lim_{k \rightarrow \infty} \frac{1}{k 2^k} = 0$ ■

- If $|r| \geq 1$, then the sequence (r^n) does not converge to 0 (L8)
 $\Rightarrow \sum_{n=0}^{\infty} r^n$ does not converge
- Consider $\sum_{n=1}^{\infty} \frac{1}{n}$: $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$, but the series diverges (L8)

Comparison test

Thm 14.6 Let (a_n) and (b_n) be two sequences, $\forall n \ a_n \geq 0$

Then

$$(i) \left(\sum_{n=1}^{\infty} a_n \text{ converges} \wedge \forall n \ (|b_n| \leq a_n) \right) \Rightarrow \sum_{n=1}^{\infty} b_n \text{ converges}$$

$$(ii) \left(\sum_{n=1}^{\infty} a_n = +\infty \wedge \forall n \ (b_n \geq a_n) \right) \Rightarrow \sum_{n=1}^{\infty} b_n = +\infty$$

Proof. (i) Use the Cauchy criterion $\left[\forall \varepsilon > 0 \exists N \forall n > m > N \left| \sum_{k=m+1}^n b_k \right| < \varepsilon \right]$

$$\left| \sum_{k=m+1}^n b_k \right| \stackrel{\text{Tr. Ineq.}}{\leq} \sum_{k=m+1}^n |b_k| \leq \sum_{k=m+1}^n a_k$$

Fix $\varepsilon > 0$. By Thm 14.4 $\exists N \forall n > m > N \sum_{k=m+1}^n a_k < \varepsilon$. Then

$$\forall n > m > N \left| \sum_{k=m+1}^n b_k \right| < \sum_{k=m+1}^n a_k < \varepsilon \quad \text{By Thm 14.4 } \sum_{n=1}^{\infty} b_n \text{ converges}$$

(ii) Denote $s_n = \sum_{k=1}^n a_k$, $t_n = \sum_{k=1}^n b_k$. Then $\forall n \ (t_n \geq s_n)$

$$\sum_{n=1}^{\infty} a_n = +\infty \Leftrightarrow \lim_{n \rightarrow \infty} s_n = +\infty \Rightarrow \lim_{n \rightarrow \infty} t_n = +\infty \Leftrightarrow \sum_{n=1}^{\infty} b_n = +\infty$$