# MATH 142A: Introduction to Analysis

math-old.ucsd.edu/~ynemish/teaching/142a

## Today: Series

> Q&A: February 4

## Next: Ross § 17

Week 5:

- Homework 4 (due Sunday, February 6)

## Comparison test

**Thm 14.6** Let $(a_n)$ and $(b_n)$ be two sequence, $\forall n\ a_n \geq 0$

Then

(i) $\left( \sum_{n=1}^{\infty} a_n \text{ converges} \wedge \forall n^{\geq N} \left( |b_n| \leq a_n \right) \right) \Rightarrow \sum_{n=1}^{\infty} b_n$ converges

(ii) $\left( \sum_{n=1}^{\infty} a_n = +\infty \quad \wedge \quad \forall n^{\geq N} \left( b_n \geq a_n \right) \right) \Rightarrow \sum_{n=1}^{\infty} b_n = +\infty$

**Examples**

- $\sum_{n=1}^{\infty} \frac{n}{3^n}$ : $\forall n \quad n \leq 2^n \Rightarrow \forall\ \frac{n}{3^n} \leq \left(\frac{2}{3}\right)^n$, $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ converges $\Rightarrow \sum_{n=1}^{\infty} \frac{n}{3^n}$ converges

- $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ : $\forall n \quad \frac{1}{n+\sqrt{n}} \geq \frac{1}{2n}$, $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$ diverges $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ diverges

**Corollary 14.7** Absolutely convergent series are convergent

Proof: $\sum x_n$ a.c. $\Rightarrow \sum_{n=1}^{\infty} \underbrace{|x_n|}_{a_n}$ converges, $\forall n\ \underbrace{|x_n|}_{|b_n|} \leq \underbrace{|x_n|}_{a_n}$ $\quad a_n = |x_n|$, $b_n = x_n \Rightarrow \underbrace{\sum x_n}_{\sum b_n}$ conv. ∎

# Root Test

**Thm 14.9** Let $\sum_{n=1}^{\infty} a_n$ be a series, let $\alpha = \limsup \sqrt[n]{|a_n|}$. Then

(i) $\alpha < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ is absolutely convergent

(ii) $\alpha > 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ diverges

(iii) $\alpha = 1$ does not provide information about the convergence of $\sum_{n\geq 1}^{\infty} a_n$

**Proof:** (i) $\alpha < 1 \Rightarrow \exists\ \beta > 0$ s.t. $\alpha < \beta < 1$

$$\limsup \sqrt[n]{|a_n|} = \alpha < \beta \Rightarrow \exists\ N_0 \quad \sup\{\sqrt[n]{|a_n|} : n > N_0\} < \beta$$

$$\Rightarrow \forall\ n > N_0 \quad \sqrt[n]{|a_n|} < \beta \Rightarrow \forall n > N_0 \quad |a_n| < \beta^n$$

Fix $\varepsilon > 0$. Since $\beta < 1$, $\exists\ N\ \forall\ n > m > N \quad \sum_{k=m+1}^{n} \beta^k < \varepsilon$

Then $\forall\ n > m > \max\{N_0, N\} \quad \sum_{k=m+1}^{n} |a_k| < \sum_{k=m+1}^{n} \beta^k < \varepsilon \Rightarrow \sum_{n=1}^{\infty} |a_n|$ converges

(ii) $\exists\ (n_k)$ s.t. $\lim_{k\to\infty} \sqrt[n_k]{|a_{n_k}|} = \alpha > 1 \Rightarrow \exists\ N\ \forall k > N \quad \sqrt[n_k]{|a_{n_k}|} > 1$

$\Rightarrow \forall k > N \quad |a_{n_k}| > 1 \Rightarrow (a_n)$ does not converge to zero $\Rightarrow \sum a_n$ diverge ∎

# Ratio Test

**Thm 14.8** Let $\sum_{n=1}^{\infty} a_n$ be a series, $\forall n\ (a_n \neq 0)$.

(i) $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \Rightarrow \sum a_n$ converges absolutely

(ii) $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1 \Rightarrow \sum a_n$ diverges

(iii) $\liminf \left|\frac{a_{n+1}}{a_n}\right| \le 1 \le \limsup \left|\frac{a_{n+1}}{a_n}\right|$ : not enough information.

**Proof** Let $\alpha = \limsup \sqrt[n]{|a_n|}$. Then by Thm 12.2

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \le \limsup \sqrt[n]{|a_n|} \le \limsup \left|\frac{a_{n+1}}{a_n}\right|.$$

(i) Follows from Thm 14.9 (i) ($\alpha < 1$)

(ii) Follow from Thm 14.9 (ii) ($\alpha > 1$)

(iii) $\sum \frac{1}{n}$, $\sum \frac{1}{n^2}$

$\blacksquare$

# Examples

- $\forall \ \alpha > 1 \quad \sum_{n=1}^{\infty} \frac{\alpha^n}{n!}$ converges

Ratio test: $\lim_{n\to\infty} \frac{\alpha^{n+1}}{(n+1)!} \cdot \frac{n!}{\alpha^n} = \lim_{n\to\infty} \frac{\alpha}{n+1} = 0 < 1$

$\Rightarrow$ by Thm 14.8 $\sum_{n=1}^{\infty} \frac{\alpha^n}{n!}$ converges

- $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges
- $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

Ratio test: $\lim_{n\to\infty} \frac{1}{(n+1)^2} \cdot n^2 = 1$ $\qquad$ $\lim_{n\to\infty} \frac{1}{n+1} \cdot n = 1$

Root test:

$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n} \cdot \sqrt[n]{n}} = 1$ $\qquad$ $\lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1$

# Integral test

- $a_n = \frac{1}{n^2}$ , $S_k = \sum_{n=1}^{k} \frac{1}{n^2}$

$\forall k \quad S_k \leq 1 + \int_1^k \frac{1}{x^2}\,dx = 1 + 1 - \frac{1}{k} < 2$

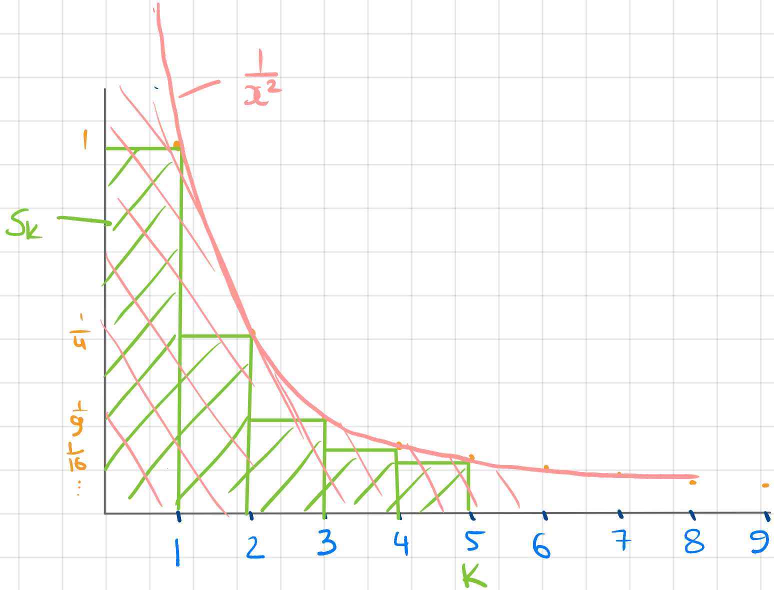
$\Rightarrow (S_k)$ increasing and bounded $\overset{\text{Thm } 10.2}{\Longrightarrow} \sum_{n=1}^{\infty} a_n$ converges

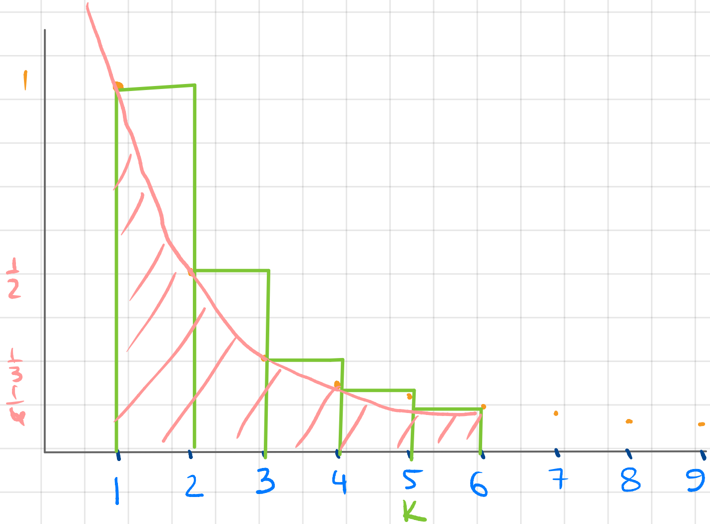
- $b_n = \frac{1}{n}$ , $t_k = \sum_{n=1}^{k} \frac{1}{n}$

$\forall k \quad t_k \geq \int_1^{k+1} \frac{1}{x}\,dx = \log(k+1)$

$\lim_{k\to\infty} \log(k+1) = +\infty \Rightarrow \sum_{n=1}^{\infty} b_n$ diverges

- $p>0$: $\sum \frac{1}{n^p}$ , $\lim_{k\to\infty} \int_1^k \frac{1}{x^p}\,dx < +\infty$ iff $p>1$



# Examples

- $a_n = \frac{1}{n \log n}$, $n \geq 3$, $\sum_{n\geq 3}^{\infty} \frac{1}{n\log n}$ [use $\forall n \geq 3 \quad 1 \leq \log n \leq n$]

Root test:

$$\sqrt[n]{\frac{1}{n^2}} \leq \sqrt[n]{\frac{1}{n\log n}} \leq \sqrt[n]{\frac{1}{n}}$$

$$\Big\downarrow n\to\infty \qquad \Big\downarrow n\to\infty \qquad \Big\downarrow n\to\infty$$

$$1 \qquad\qquad 1 \qquad\qquad 1$$

$$\int_3^k \frac{1}{x\log x}\,dx = \log\log k - \log\log 3 \qquad \left(\log(\log(x))\right)' = \frac{1}{\log x}\cdot\frac{1}{x}$$

$$\Big\downarrow k\to\infty$$

$$+\infty$$

$$\Rightarrow \sum_{n\geq 3}^{\infty} \frac{1}{n\log n} \text{ diverges}$$

# Alternating Series

**Thm 15.3** Let $(a_n)$ be a sequence s.t. $\forall n\ (a_n \geq 0 \wedge a_n \geq a_{n+1})$. Then

$$\lim_{n\to\infty} a_n = 0 \Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} a_n \text{ converges and } \forall n \left| \sum_{k=1}^{\infty} (-1)^{k+1} a_k - \sum_{k=1}^{n} (-1)^{k+1} a_k \right| \leq a_n$$

**Proof.** Denote $\sum_{k=1}^{n} (-1)^{k+1} a_k =: S_n$.

① $(S_{2n})_{n=1}^{\infty}$ is increasing, $(S_{2n-1})_{n=1}^{\infty}$ is decreasing

$$S_{2n} - S_{2(n-1)} = a_{2n-1} - a_{2n} \geq 0, \quad S_{2n+1} - S_{2n-1} = -a_{2n} + a_{2n+1} \leq 0$$

② $\forall m, n \in \mathbb{N}\ (S_{2m} \leq S_{2n+1})$

Case $m \leq n$: $S_{2m} \leq S_{2n} \leq S_{2n} + (-1)^{2n+2} a_{2n+1} = S_{2n+1}$

Case $m \geq n$: $S_{2m} \leq S_{2m} + (-1)^{2m+2} a_{2m+1} = S_{2m+1} \leq S_{2n+1}$

By ② + Thm 10.2 $\lim_{n\to\infty} S_{2n} =: S_2 \leq S_1 := \lim_{n\to\infty} S_{2n-1}$ and $S_2 - S_1 = \lim_{n\to\infty} (S_{2n} - S_{2n-1}) = \lim_{n\to\infty} a_{2n} = 0$

$\Rightarrow S_1 = S_2 =: S$ Then $\forall n\ (S_{2n} \leq S \leq S_{2n+1}) \Rightarrow \forall n\ \max\{|S - S_{2n}|, |S - S_{2n+1}|\} \leq S_{2n+1} - S_{2n}$

$= a_{2n+1} \leq a_{2n}$ ∎

## Important example

9. Let $p>0$. Then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p>1$

Proof. Denote $x_n = \frac{1}{n^p}$, $S_k = \sum_{n=1}^{k} x_n$. $x_1 \geq x_2 \geq \cdots \geq x_n$, $(S_k)$ is increasing.

Consider the sequences: $a_1 = x_1$, $a_2 = 2\cdot x_2$, $a_3 = 4\cdot x_4, \dots, a_k = 2^{k-1}\cdot x_{2^{k-1}}$

$b_1 = x_2$, $b_2 = 2\cdot x_4$, $b_3 = 4\cdot x_8, \dots, b_k = 2^{k-1}\cdot x_{2^k}$

Then $b_1 \leq x_2 \leq a_1$, $b_2 \leq x_3 + x_4 \leq a_2$, $b_3 \leq x_5 + x_6 + x_7 + x_8 \leq a_3$

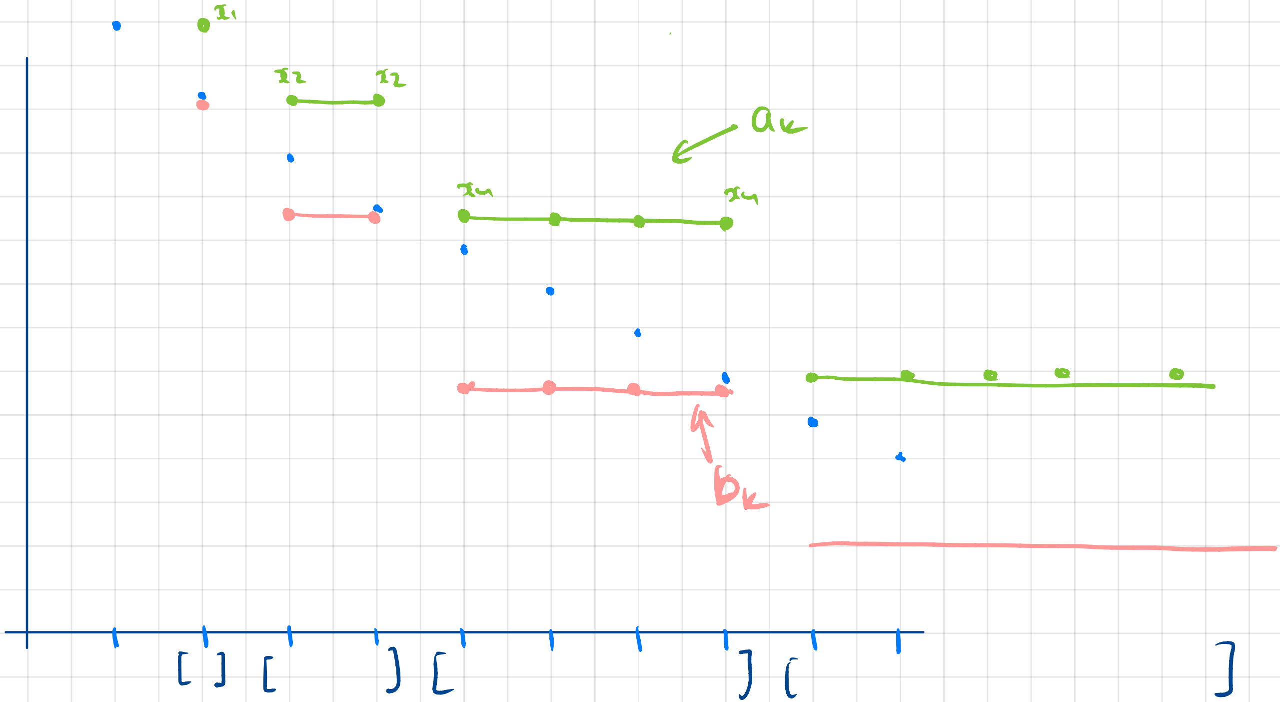
$b_k \leq x_{2^{k-1}+1} + \cdots + x_{2^k} \leq a_k$

and $\forall k$ (a) $x_1 + \sum_{n=1}^{k} b_n \leq S_{2^k} \leq x_1 + \underbrace{\sum_{n=1}^{k} a_n}_{A_k}$; (b) $\sum_{n=1}^{k} b_n = \frac{1}{2}\sum_{n=1}^{k} a_{n+1}$

① $(S_k)$ converges $\Leftrightarrow$ $(S_{2^k})$ converges ($\Rightarrow$ Thm 11.3; $\Leftarrow$ Thm 9.1 + $(S_k \leq S_{2^k})$ + Thm 10.2)

② $(S_{2^k})$ converges $\overset{(a),(b)}{\Leftrightarrow}$ $(A_k)$ converges $\Leftrightarrow$ $\sum a_n$ converges

$a_n = 2^{n-1}\cdot x_{2^{n-1}} = 2^{n-1}\cdot\frac{1}{(2^{n-1})^p} = 2^{(1-p)\cdot(n-1)} = \frac{1}{2^{1-p}}\left(2^{(1-p)}\right)^n$. By I.E8, $\sum a_n$ converges iff $2^{1-p} < 1 \Leftrightarrow 1-p<0 \Leftrightarrow p>1$. ∎



x1
x2
x2
ak
x4
x4
bk