MATH 142A: Introduction to Analysis

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Today: Continuous functions

> Q&A: February 7

Next: Ross § 18

Week 6:

- Homework 5 (due Sunday, February 13)
- Homework 3 regrades Tuesday, February 8

Functions

Def. (Function) Let X and Y be two sets. We say that there is a function defined on X with values in Y, if via some rule f we associate to each element $x \in X$ an (one) element $y \in Y$. We write $f: X \to Y$, $x \overset{f}{\mapsto} y$ (or $y = f(x)$).

X is called the domain of definition of the function, $\mathrm{dom}(f)$, $y = f(x)$ is called the image of x. $f: [0,1) \to [0,1)$, $x \mapsto x^2$

Remarks 1) We consider real-valued functions ($Y \subset \mathbb{R}$) of one real variable ($X \subset \mathbb{R}$).

2) If $\mathrm{dom}(f)$ is not specified, then it is understood that we take the natural domain: the largest subset of $\mathbb{R}$ which the function is well defined

$$\left(\begin{array}{ll} f(x) = \sqrt{x} \text{ means } \mathrm{dom}(f) = [0, +\infty) \\ g(x) = \dfrac{1}{x^2 - x} \text{ means } \mathrm{dom}(g) = \mathbb{R} \setminus \{0, 1\} \end{array} \right)$$

Continuity of a function at a point

Intuitively: Function f is continuous at point $x_0 \in \mathrm{dom}(f)$ if

$f(x)$ approaches $f(x_0)$ as x approaches x_0.

Def 17.1 (Continuity). Let f be a real-valued function, $\mathrm{dom}(f) \subset \mathbb{R}$.

Function f is continuous at $x_0 \in \mathrm{dom}(f)$ if for any sequence

(x_n) in $\mathrm{dom}(f)$ converging to x_0, we have

Def 17.6 (Continuity) Let f be a real-valued function.

Function f is continuous at $x_0 \in \mathrm{dom}(f)$ if

Remark Def 17.1 is called the sequential definition of continuity,

Def 17.6 is called the ε-δ definition of continuity.

Equivalence of sequential and ε-δ definitions

Thm 17.2. Definitions 17.1 and 17.6 are

Proof ($17.1 \Rightarrow 17.6$). Suppose that (*) fails

$$\forall \varepsilon>0 \ \exists \delta>0 \ \left(x\in \mathrm{dom}(f) \wedge |x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|<\varepsilon \right) \quad (*)$$

This means that

Take $\delta =$:

$\Rightarrow$

($\Leftarrow$). Let (x_n) be such that $\lim x_n = x_0$. Fix $\varepsilon>0$. By (*)

$\lim_{n\to\infty} x_n = x_0 \Rightarrow$ Therefore

$$\forall n>N \ (x_n \in \mathrm{dom}(f) \wedge |x_n - x_0|<\delta) \overset{(*)}{\Longrightarrow}$$

Continuity on a set. Examples

Def 17.1 Let f be a function, and let $S \subset \operatorname{dom}(f)$.

f is continuous on S if for all $x_0 \in S$ f is continuous at x_0.

Example 1) $f(x) = \frac{2x}{x^2-1}$ is continuous on $\mathbb{R} \setminus \{-1, 1\}$

Proof. Let $x_0 \in \mathbb{R} \setminus \{-1,1\}$ and let (x_n) be such that $\forall n$ $x_n \notin \{-1,1\}$

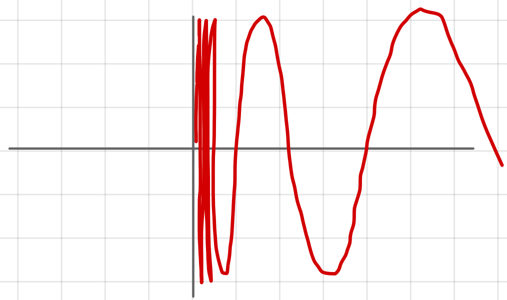
and $\lim x_n = x_0$. Then by Thm 9.2, 9.3, 9.6

$\lim f(x_n) =$

By Def 17.1 f is continuous at x_0 for any $x_0 \in \mathbb{R} \setminus \{-1,1\}$

2) $g(x) = \sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and $g(0) = a$. Then for any $a \in \mathbb{R}$

g is not continuous at 0.



Proof Take (x_n) with $x_n =$

Then and

$\Rightarrow$

Continuity and arithmetic operations

Thm 17.3 Let f be a real-valued function with $\mathrm{dom}(f) \subset \mathbb{R}$.

If f is continuous at $x_0 \in \mathrm{dom}(f)$, then

Proof. Let (x_n) be a sequence in $\mathrm{dom}(f)$ such that $\lim_{n\to\infty} x_n = x_0$.

Then by Thm 9.2

Therefore $k \cdot f$ is continuous at x_0.

By the triangle inequality

Fix $\varepsilon > 0$. Then $\lim f(x_n) = f(x_0) \Rightarrow$

Then $\forall n > N$

This means that $\lim_{n\to\infty} |f(x_n)| = |f(x_0)|$, $|f|$ is continuous at x_0.

Continuity and arithmetic operations

<u>Thm 17.4</u> Let f and g be real-valued functions that are continuous at $x_0 \in \mathbb{R}$. Then

(i) $f+g$ is continuous at x_0 (ii) $f \cdot g$ is continuous at x_0

(iii) if $g(x_0) \neq 0$, then $\frac{f}{g}$ is continuous at x_0.

<u>Proof:</u> Note that if $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$, then $(f+g)(x) = f(x) + g(x)$ and $f \cdot g(x) = f(x) \cdot g(x)$ are well-defined. Moreover, if $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$ and $g(x) \neq 0$, then $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$ is well-defined.

Let (x_n) be a sequence in $\mathrm{dom}(f) \cap \mathrm{dom}(g)$ s.t. $\lim x_n = x_0$.

Then $\lim (f(x_n) + g(x_n)) =$, and

$\lim (f(x_n) \cdot g(x_n)) =$. If moreover $\forall n \; g(x_n) \neq 0$

then $\lim \frac{f(x_n)}{g(x_n)} =$

Continuity of a composition of functions

Let f and g be real-valued functions. If $x \in \mathrm{dom}(f)$ and $f(x) \in \mathrm{dom}(g)$, then we define

Thm 17.5 If f is continuous at x_0 and g is continuous at $f(x_0)$, then

Proof It is given that $x_0 \in \mathrm{dom}(f)$ and $f(x_0) \in \mathrm{dom}(g)$. Let (x_n) be a sequence such that and $\lim x_n = x_0$. Denote . Since f is continuous at x_0, $\lim y_n =$ Since g is continuous at $f(x_0) = y_0$, we have $\lim g \circ f(x_n) =$ Therefore, $g \circ f$ is continuous at x_0.

Examples

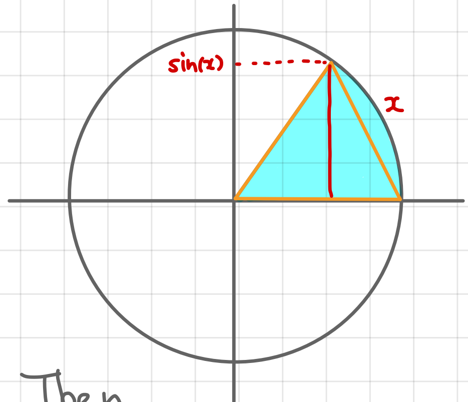
1) $\sin(x)$ is continuous on $\mathbb{R}$

<u>Proof</u> ① Enough to show that $\sin(x)$ is continuous at 0

For any $x_0 \in \mathbb{R}$ and (x_n) with $\lim x_n = x_0$

$$|\sin(x_n) - \sin(x_0)| =$$

② Area $(\triangle) \leq$ Area $(\triangle)$



$\Rightarrow \forall\ x \in [0, \frac{\pi}{2}]$

$\Rightarrow$

③ If $\lim y_n = 0$, then

$\forall\ n > N$

Then

Examples

2) $f(x) = \sqrt{x}$ is continuous on $[0, +\infty)$.

① $\sqrt{x}$ is continuous at 0

Let $\lim x_n = 0$. Fix $\varepsilon > 0$. Then

$\Rightarrow$

② Let $x_0 \in (0, +\infty)$, (x_n) s.t. $\forall n\ (x_n \in [0, +\infty))$ and $\lim x_n = x_0$

Then $\lim x_n = x_0 > 0 \overset{T9.11(i)}{\Rightarrow}$

Fix $\varepsilon > 0$. Then . Then

$\forall n > \max\{N_1, N_2\}\ |f(x_n) - f(x_0)| = |\sqrt{x_n} - \sqrt{x_0}| =$

3) $\cos(x)$ is continuous on $\mathbb{R}$. $\cos(x) =$, by Thm 17.4

is continuous on $\mathbb{R}$. Moreover, $\forall x \in \mathbb{R}$

$\Rightarrow$ by example 2) and Thm 17.5