

MATH 142A: Introduction to Analysis

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Today: Properties of continuous functions

> Q&A: February 9

Next: Ross § 19

Week 6:

- Homework 5 (due Sunday, February 13)
- Homework 3 regrades Tuesday, February 8

The maximum-value theorem

Def. 18.7 Let f be a function and let $A \subset \text{dom}(f)$. f is called bounded on A if $\exists M > 0 \quad \forall x \in A \quad |f(x)| \leq M$

Thm 18.1 Let f be a function, $[a, b] \subset \text{dom}(f)$, f is continuous on $[a, b]$

Then (i) f is bounded on $[a, b]$

(ii) $\exists x_0, y_0 \in [a, b]$ s.t. $\forall x \in [a, b] \quad (f(x_0) \leq f(x) \leq f(y_0))$
 \uparrow min value \uparrow max value

Proof (i) Suppose that f is not bounded on $[a, b]$

$$\Rightarrow \forall n \in \mathbb{N} \quad \exists x_n \in [a, b] \quad \text{s.t.} \quad |f(x_n)| > n \quad (*)$$

① (x_n) is bounded $\stackrel{\text{T.11.5}}{\Rightarrow} \exists (x_{n_k})$ s.t. (x_{n_k}) converges

② $\forall k \quad a \leq x_{n_k} \leq b \stackrel{\text{T.9.11}}{\Rightarrow} \lim x_{n_k} =: \tilde{x} \in [a, b]$

③ $\tilde{x} \in [a, b]$, f cont. on $[a, b] \stackrel{\text{T.17.3}}{\Rightarrow} |f|$ is cont. at \tilde{x}

$\Rightarrow \lim |f(x_{n_k})| = |f(\tilde{x})| \Rightarrow \exists N \quad \forall k > N \quad (|f(x_{n_k})| < |f(\tilde{x})| + 1)$
contradiction to $(*)$.

The maximum-value theorem

Proof (ii) Denote $M := \sup\{f(x) : x \in [a, b]\}$. By (i), $M < +\infty$

① $M = \sup\{f(x) : x \in [a, b]\} \Rightarrow \forall n \exists x_n \in [a, b] \left(M - \frac{1}{n} < f(x_n) \leq M\right)$

② $\forall n \left(M - \frac{1}{n} < f(x_n) \leq M\right) \stackrel{T.9.11}{\Rightarrow} \lim f(x_n) = M$

③ $\forall n (x_n \in [a, b]) \stackrel{T.11.5}{\Rightarrow} \exists (x_{n_k}) \exists y_0 \in [a, b] \text{ s.t. } \lim x_{n_k} = y_0$

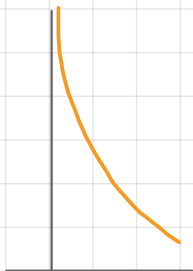
④ $y_0 \in [a, b] \Rightarrow f \text{ is continuous at } y_0 \Rightarrow \lim f(x_{n_k}) = f(y_0)$
 $\Rightarrow \text{by T.11.3 } f(y_0) = M \Rightarrow \forall x \in [a, b] (f(x) \leq f(y_0))$

(Exercise: prove that $\exists x_0 \in [a, b]$ s.t. $\forall x \in [a, b] (f(x_0) \leq f(x))$)

Examples

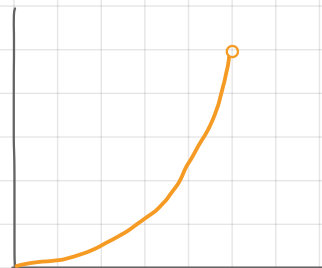
1) $f(x) = \frac{1}{x}$

- continuous on $(0, 1]$
- unbounded on $(0, 1]$



2) $f(x) = x^2$

- cont on $[0, 1]$
- no maximum on $[0, 1]$



Intermediate value theorem

Thm 18.2 Let f be continuous on the interval $I \subset \mathbb{R}$. Let $a, b \in I$
s.t. $a < b$. Then

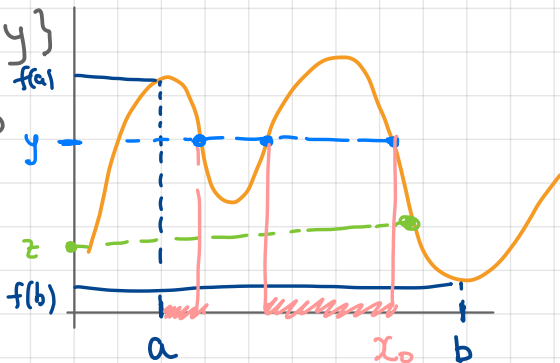
(i) $f(a) < f(b)$ and $y \in (f(a), f(b)) \Rightarrow \exists x \in (a, b)$ s.t. $f(x) = y$

(ii) $f(a) > f(b)$ and $y \in (f(b), f(a)) \Rightarrow \exists x \in (a, b)$ s.t. $f(x) = y$

Proof (ii) Consider $S = \{x \in [a, b] : f(x) > y\}$

① $a \in S$, $\sup S \geq a$, $\sup(S) \leq \sup[a, b] = b$
 $\Rightarrow \sup S =: x_0 \in [a, b]$

② $\forall n \in \mathbb{N} \exists s_n \in S$ s.t. $x_0 - \frac{1}{n} < s_n \leq x_0$



Then $\lim s_n = x_0 \wedge f(s_n) > y \wedge f$ cont at $x_0 \Rightarrow f(x_0) = \lim f(s_n) \geq y$

③ Define $t_n := \min\{x_0 + \frac{1}{n}, b\}$. $\forall n$ $t_n \in [a, b] \setminus S \Rightarrow \forall n$ $f(t_n) \leq y$
 $\lim t_n = x_0 \Rightarrow f(x_0) = \lim f(t_n) \leq y \stackrel{\text{②+③}}{\Rightarrow} f(x_0) = y \Rightarrow x_0 \in (a, b)$

Image of an interval

Cor. 18.3 Let f be continuous on the interval I . Then

$f(I) = \{ f(x) : x \in I \}$ is an interval or a single point.

Proof If $\forall x \in I f(x) = y_0$, then $f(I) = y_0$.

① Let $y_1 < y_2 \in f(I)$. Then $\exists x_1, x_2 \in I$ s.t. $f(x_1) = y_1, f(x_2) = y_2$

Let $y \in (y_1, y_2)$.

• If $x_1 < x_2$, then by Thm 18.2 (i) $\exists x_0 \in (x_1, x_2) \subset I$ s.t. $f(x_0) = y$
 $\Rightarrow y \in f(I)$

• If $x_2 < x_1$, then by Thm 18.2 (ii) $\exists x_0 \in (x_2, x_1) \subset I$ s.t. $f(x_0) = y$
 $\Rightarrow y \in f(I)$.

② Let $\inf f(I) < \sup f(I)$. Then $\forall y \in (\inf f(I), \sup f(I)) \exists y_1, y_2 \in f(I)$
s.t. $y_1 < y < y_2 \Rightarrow y \in f(I) \Rightarrow f(I) \supset (\inf f(I), \sup f(I))$ ■

Examples

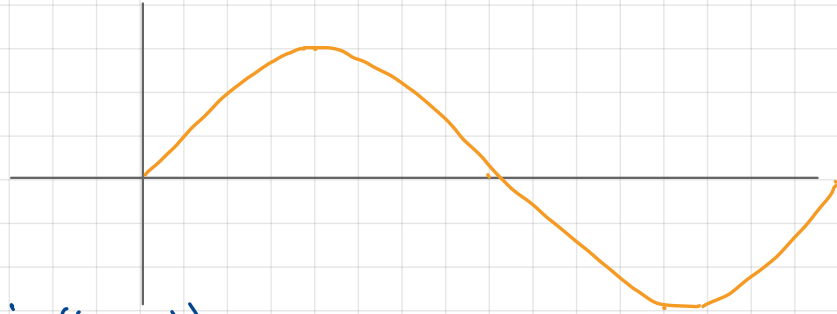
$$1) \sin : (0, 2\pi) \rightarrow \mathbb{R}$$

$$\sin((0, 2\pi)) \subset [-1, 1]$$

$$\sin\left(\frac{\pi}{2}\right) = 1$$

$$\sin\left(\frac{3\pi}{2}\right) = -1$$

$$\Rightarrow [-1, 1] \subset \sin((0, 2\pi))$$



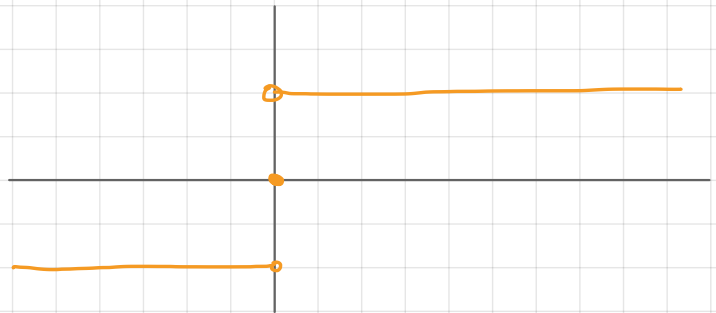
$$2) f : [-1, 1] \rightarrow \mathbb{R}, f(x) = \operatorname{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

$$f([-1, 1]) = \{-1, 0, 1\}$$

$$-1 = f\left(-\frac{1}{2}\right) < f(0) = 0$$

But $\forall y \in (-1, 0)$

$$\left\{ x \in \left(-\frac{1}{2}, 0\right) : f(x) = y \right\} = \emptyset$$



Continuity of strictly increasing functions.

Def 18.8 Function f is called

(strictly) increasing if $x < y \Rightarrow f(x) \leq f(y)$ ($f(x) < f(y)$)

(strictly) decreasing if $x < y \Rightarrow f(x) \geq f(y)$ ($f(x) > f(y)$)

Thm 18.5 Let g be strictly increasing function on interval J .

If $g(J)$ is an interval, then g is continuous of J

Proof. Let $x_0 \in J$, $x_0 > \inf J$, $x_0 < \sup J$. Then $g(x_0) > \inf g(J)$

$g(x_0) < \sup g(J) \Rightarrow \exists \varepsilon_0 > 0$ s.t. $(g(x_0) - \varepsilon_0, g(x_0) + \varepsilon_0) \subset g(J)$

Verify the ε - δ definition of continuity. Fix $\varepsilon > 0$, $\varepsilon < \varepsilon_0$.

Then $\exists x_1, x_2 \in J$ s.t. $g(x_1) = g(x_0) - \varepsilon$, $g(x_2) = g(x_0) + \varepsilon$, $x_1 < x_0 < x_2$

Now, $\forall x \in (x_1, x_2)$ $g(x_1) < g(x) < g(x_2) \Rightarrow |g(x) - g(x_0)| < \varepsilon$

Take $\delta = \min\{x_0 - x_1, x_2 - x_0\}$ Then $|x - x_0| < \delta \Rightarrow x \in (x_1, x_2)$

$\Rightarrow |g(x) - g(x_0)| < \varepsilon$ ■

Inverse function

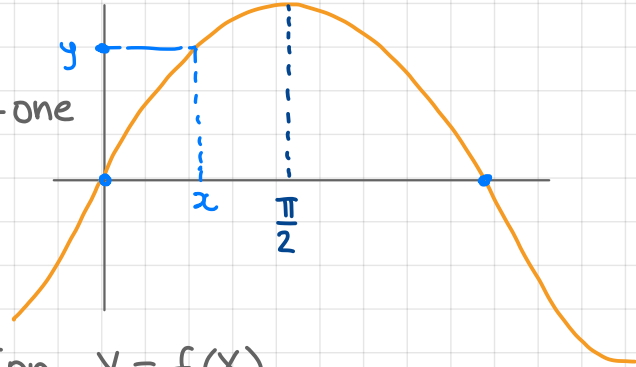
Def 18.9 Function $f: X \rightarrow Y$ is called **one-to-one (or bijection)**
 $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

if $f(x) = y$ and $\forall y \in Y \exists! x \in X$ s.t. $f(x) = y$

Example $\sin: [-\frac{\pi}{2}, \frac{\pi}{2}] \rightarrow [-1, 1]$ is one-to-one

$\sin: [0, \pi] \rightarrow [0, 1]$ is not one-to-one

$$\sin(0) = \sin(\pi) = 0$$



Def 18.10 Let $f: X \rightarrow Y$ be a bijection, $Y = f(X)$.

Then the function $f^{-1}: Y \rightarrow X$ given by $(f^{-1}(y) = x \Leftrightarrow f(x) = y)$ is called the **inverse of f** . In particular $f^{-1}(f(x)) = x, f(f^{-1}(y)) = y$

Example • $\sin: [-\frac{\pi}{2}, \frac{\pi}{2}] \rightarrow [-1, 1], \sin^{-1} = \arcsin: [-1, 1] \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$

• $f: [0, +\infty) \rightarrow [0, +\infty), f(x) = x^m, f^{-1}: [0, +\infty) \rightarrow [0, +\infty), f^{-1}(x) = x^{\frac{1}{m}} = \sqrt[m]{x}$

• If f is strictly increasing (decreasing) on X , then $f: X \rightarrow f(X)$ is a bijection

Continuity and the inverse function

Thm 18.4 Let f be a continuous strictly increasing function on some interval I . Then $J := f(I)$ is an interval and $f^{-1}: J \rightarrow I$ is continuous and strictly increasing.

Proof ① f^{-1} is strictly increasing: Take $y_1, y_2 \in J$, $y_1 < y_2$

Denote $x_1 = f^{-1}(y_1)$, $x_2 = f^{-1}(y_2)$. Then $f(x_1) = y_1$, $f(x_2) = y_2$

If $x_1 \geq x_2$, then $f(x_1) \geq f(x_2)$, contradiction $\Rightarrow x_1 < x_2$

② J is an interval: By Cor. 18.3 J is either an interval or a single point. Since f is strictly increasing, J is an interval

③ ① + ② + Thm 18.5 f^{-1} is continuous on J .

One-to-one continuous functions

Thm 18.6 Let f be a one-to-one continuous function on an interval I . Then f is strictly increasing or strictly decreasing on I .

Proof. ① If $a < b < c$ then either $f(a) < f(b) < f(c)$ or $f(c) < f(b) < f(a)$

Otherwise, $f(b) > \max\{f(a), f(c)\}$ or $f(b) < \min\{f(a), f(c)\}$

If $f(b) > \max\{f(a), f(c)\}$, choose $y \in (\max\{f(a), f(c)\}, f(b))$

Then by Thm 18.2 $\exists x_1 \in (a, b)$ s.t. $f(x_1) = y$, $\exists x_2 \in (b, c)$ s.t. $f(x_2) = y$

\Rightarrow contradiction Similarly when $f(b) < \min\{f(a), f(c)\}$.

② Take any $a_0 < b_0$. If $f(a_0) < f(b_0)$, then f is increasing on I .

$$x < a_0 < y < b_0 < z \begin{array}{l} \textcircled{1} \\ \Rightarrow f(a_0) < f(y) < f(b_0) \end{array} \left| \begin{array}{l} \Rightarrow f(x) < f(a_0) < f(y) \\ \textcircled{1} \\ \Rightarrow f(y) < f(b_0) < f(z) \end{array} \right.$$
$$\Rightarrow \forall x_1 < x_2 \quad (f(x_1) < f(x_2))$$

③ Similarly, if $f(a_0) > f(b_0)$, then f is decreasing. ■

