

MATH 142A: Introduction to Analysis

math-old.ucsd.edu/~ynemish/teaching/142a

Today: Uniform continuity

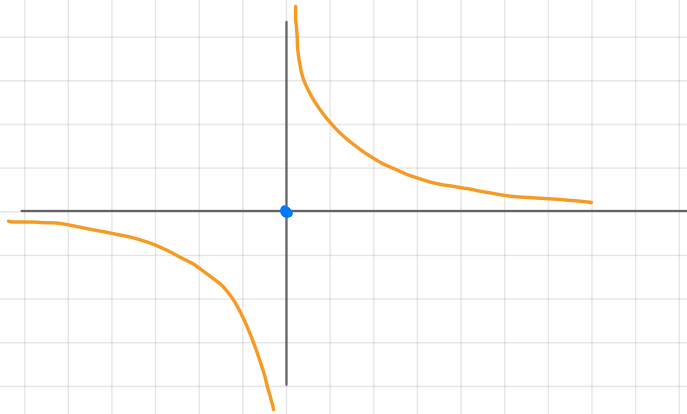
> Q&A: February 14

Next: Ross § 20

Week 7:

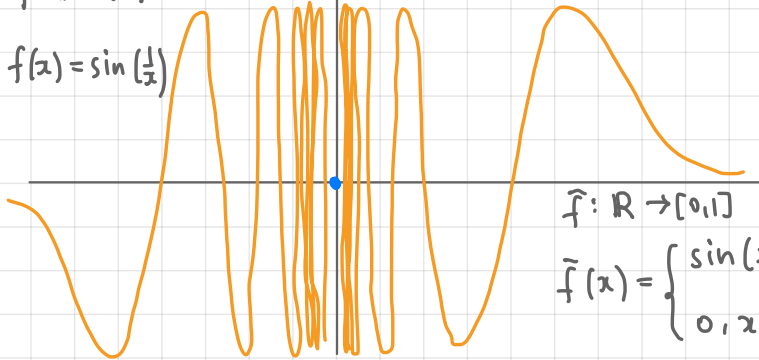
- Homework 6 (due Sunday, February 20)

Extension of a function



$$f: \mathbb{R} \setminus \{0\} \rightarrow [0, 1]$$

$$f(x) = \sin\left(\frac{1}{x}\right)$$

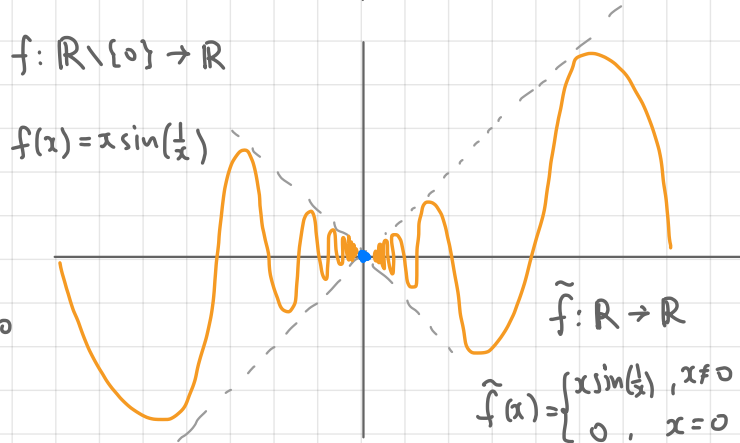


$$\tilde{f}: \mathbb{R} \rightarrow [0, 1]$$

$$\tilde{f}(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$f: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$$

$$f(x) = x \sin\left(\frac{1}{x}\right)$$



$$\tilde{f}: \mathbb{R} \rightarrow \mathbb{R}$$

$$\tilde{f}(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Def. 19.7. Let f and \tilde{f} be two functions s.t.

We say that \tilde{f} is an **extension** of f if

Continuous extension

Thm 19.5 A real-valued function f on (a,b) is uniformly continuous on (a,b) if and only if

Proof (\Leftarrow) \tilde{f} is cont. on $[a,b] \stackrel{T 19.2}{\Rightarrow} \tilde{f}$ is unif. cont. on $[a,b]$
 $\Rightarrow \tilde{f}$ is unif. cont. on $(a,b) \Rightarrow f$ is unif. cont. on (a,b) .

(\Rightarrow) Suppose f is unif. cont. on (a,b) .

① Let (s_n) be a sequence, $s_n \in (a,b)$, $\lim s_n = a$.

(s_n) converges

② Let (s_n) and (t_n) be two sequences, $\forall n s_n, t_n \in (a,b)$, $\lim s_n = \lim t_n = a$

Take

Then $u_n \in (a,b)$, $\lim u_n = a$

③ \tilde{f} is continuous at a (follows from Lemma 19.8).

Continuous extension

Lemma 19.8 (Ex. 17.15) Let f be a real-valued function whose domain is a subset of \mathbb{R} . Then f is continuous at $x_0 \in \text{dom}(f)$ iff for any sequence (x_n) in $\text{dom}(f) \setminus \{x_0\}$ converging to x_0 , we have $\lim f(x_n) = f(x_0)$

Proof (\Rightarrow) Trivial

(\Leftarrow) Let (s_n) be a sequence in $\text{dom}(f)$, $\lim s_n = x_0$.

(i) $\{n : s_n \neq x_0\}$ is finite

(ii) $\{n : s_n \neq x_0\}$ is infinite. Let (s_{n_k}) be a subsequence of (s_n) obtained by . Then (s_{n_k}) is

Fix $\varepsilon > 0$. Then

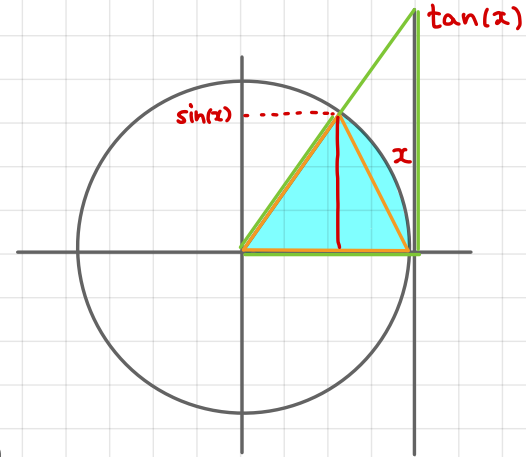
Examples

1. $f(x) = \sin\left(\frac{1}{x}\right)$ is continuous on $[-n, n] \setminus \{0\}$, but not uniformly continuous on (cannot be continuously extended to $[-n, n]$)
2. $f(x) = \frac{\sin x}{x}$ is continuous on $[-n, n] \setminus \{0\}$

$$\tilde{f}(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases} \text{ is continuous on } [-n, n] \Rightarrow f \text{ is unif. cont. on } [-n, n] \setminus \{0\}$$

Proof: Area (\triangle) \leq Area (\triangle) \leq Area (\triangle)

$$0 < |x| < \frac{\pi}{2} :$$



We want to show that \tilde{f} is cont. at $x=0$.

Fix $\varepsilon > 0$. Let (s_n) be a sequence in $[-n, n] \setminus \{0\}$,

$$\lim s_n = 0$$

Definition of some functions

\sin, \cos, \tan, \cotan

\sin, \cos are continuous on \mathbb{R}

$x^n, x \in \mathbb{R}, n \in \mathbb{N}$

x^n is continuous on \mathbb{R} for any $n \in \mathbb{N}$

x^n is a bijection from $[0, +\infty)$ to $[0, +\infty)$,

we denote the inverse by $\sqrt[n]{x} = x^{\frac{1}{n}}, x \geq 0, n \in \mathbb{N}$

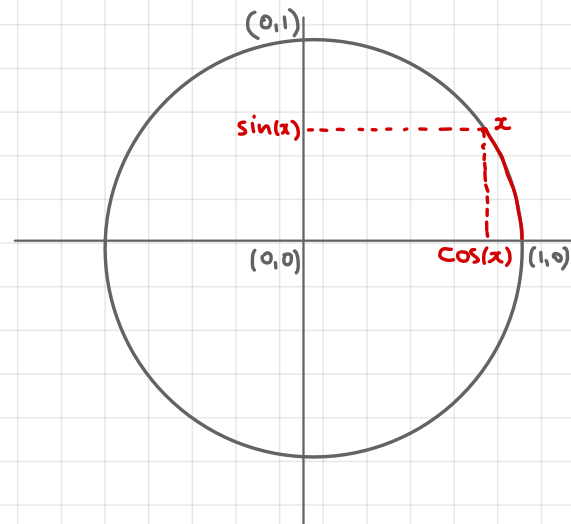
$\forall a > 0 \quad \forall m, n \in \mathbb{N} \quad (a^m)^{\frac{1}{n}} = (a^{\frac{1}{n}})^m =: a^{\frac{m}{n}}$

Let $b > 0, (q_n)$ s.t. $q_n \in \mathbb{Q} \cap (0, +\infty), q_n < q_{n+1}, \lim q_n = b$

For $a > 1$ (a^{q_n}) is increasing and bounded above $\Rightarrow \lim_{n \rightarrow \infty} a^{q_n} =: a^b > 0$

Define $(\frac{1}{a})^b = \frac{1}{a^b} = a^{-b}, a^0 = 1$

Satisfies usual properties: $a^{b_1} a^{b_2} = a^{b_1+b_2}, a^b a_2^b = (a_1 a_2)^b, \dots$



Definition of some functions

For any $a > 1$ the function $f: \mathbb{R} \rightarrow (0, \infty)$, $f(x) = a^x$ is strictly increasing, we denote the inverse by $\log_a x$

Similarly for $a \in (0, 1)$, a^x is strictly decreasing.

Usual properties hold: $\log_a x_1 + \log_a x_2 = \log_a (x_1 x_2), \dots$

Special notation: $\log_e x = \log x = \ln x$

Example of a proof: $a^{b_1} a^{b_2} = a^{b_1 + b_2}$

① If $b_1 = m_1$, $b_2 = m_2$, $m_1, m_2 \in \mathbb{N}$, then $a^{m_1} \cdot a^{m_2} = a^{m_1 + m_2}$

② If $b = \frac{1}{n}$, $a_1, a_2 \in (0, \infty)$, then $a_1^{\frac{1}{n}} \cdot a_2^{\frac{1}{n}} = (a_1 a_2)^{\frac{1}{n}}$

③ If $b_1 = \frac{m_1}{n}$, $b_2 = \frac{m_2}{n}$, then $a^{b_1} a^{b_2} = a^{b_1 + b_2}$

④ Let $(s_n), (t_n)$, $\lim s_n = b_1$, $\lim t_n = b_2$, $(s_n), (t_n)$ increasing in \mathbb{Q}
 $\forall n$ $a^{s_n} a^{t_n} = a^{s_n + t_n}$, $(s_n + t_n)$ increasing $\Rightarrow \lim a^{s_n} a^{t_n} = \lim a^{s_n + t_n}$