MATH 142A: Introduction to Analysis

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Today: Limits of functions

> Q&A: February 18

Next: Ross § 28

Week 7:

- Homework 6 (due Sunday, February 20)
- Midterm 2 (Wednesday, February 23): Lectures 8-16

Limit of a function, $\varepsilon-\delta$ definition

<u>D 20.12</u> Let f be a functions defined on $S \subset \mathbb{R}$, let $a \in \mathbb{R}$ be a limit of some sequence in S, let $L \in \mathbb{R}$. We say that f tends to L as x tends to a along S if

<u>Thm 20.6</u> Definitions 20.1 and 20.12 are equivalent.

<u>Proof</u> ($\Rightarrow$) Suppose that $(*)$ does not hold:

($\Leftarrow$) Let (x_n) be a sequence, $\forall n\ x_n \in S$, $\lim x_n = a$.

Fix $\varepsilon > 0$. Take δ as in $(*)$. $\lim x_n = a \Rightarrow$

By $(*)$ $\forall n > N$

Limit of a function, ε-δ definition

D 20.13 Suppose that f is defined on $(a-c, a+c)\setminus\{a\}$ for some $c>0$. $a, L \in \mathbb{R}$

(a) We say that L is the (two-sided) limit of f at a if

(b) We say that L is the right-hand limit of f at a if

(c) We say that L is the left-hand limit of f at a if

Corollary 20.7-20.8 Definitions 20.3 (a), (b), (c) and 20.13 (a), (b), (c) are equivalent.

Proof Follows from Thm 20.6 by specializing

(a) , (b) , (c)

Limit of a function

Suppose $f: S \to \mathbb{R}$, $a, L \in \mathbb{R}$

- $\lim\limits_{x \to +\infty} f(x) = L \overset{\text{Def}}{\iff}$

- $\lim\limits_{x \to +\infty} f(x) = +\infty \overset{\text{Def}}{\iff}$

- $\lim\limits_{x \to +\infty} f(x) = -\infty \overset{\text{Def}}{\iff}$

- $\lim\limits_{x \to -\infty} f(x) = L \overset{\text{Def}}{\iff} \forall \varepsilon > 0 \ \exists t > 0 \ (x < -t \Rightarrow |f(x) - L| < \varepsilon)$

- $\lim\limits_{x \to -\infty} f(x) = +\infty \overset{\text{Def}}{\iff} \forall M > 0 \ \exists t > 0 \ (x < -t \Rightarrow f(x) > M)$

- $\lim\limits_{x \to a} f(x) = +\infty \overset{\text{Def}}{\iff} \forall M > 0 \ \exists \delta > 0 (|x - a| < \delta \Rightarrow f(x) > M)$

- $\lim\limits_{x \to a^-} f(x) = +\infty \overset{\text{Def}}{\iff} \forall M > 0 \ \exists \delta > 0 (x \in (a - \delta, a) \Rightarrow f(x) > M)$

Two-sided limits and left-hand/right-hand limits

Thm 20.10

Let f be a function defined on $J\setminus\{a\}$ for some open interval J containing $a\in\mathbb{R}$. Let $L\in\mathbb{R}\cup\{+\infty,-\infty\}$. Then

$$\lim_{x\to a} f(x) = L \iff \lim_{x\to a^+} f(x) = L \ \wedge\ \lim_{x\to a^-} f(x) = L$$

Proof. $(\Rightarrow)$ Exercise

$(\Leftarrow)$ Suppose $L\in\mathbb{R}$. Fix $\varepsilon>0$.

Suppose $L=+\infty$. Fix $M>0$.

Examples

1)

2) Let $a>1$, $p\in\mathbb{N}$, $f:\mathbb{R}\to\mathbb{R}$, $f(x)=$. Then

Fix $\varepsilon>0$. By IE 6

Then and

Squeeze Lemma

Thm. 20.14 Let $f, g, h : S \to \mathbb{R}$, $\forall x \in S$ $f(x) \le g(x) \le h(x)$

Let $a, L \in \mathbb{R} \cup \{+\infty, -\infty\}$.

If $\lim_{S \ni x \to a} f(x) = \lim_{S \ni x \to a} h(x) = L$, then

Proof. Take any sequence (s_n) in S s.t. $\lim s_n = a$. Then

IE 12 $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$. Fix $\varepsilon > 0$. By IE from Lecture 7,

and thus

$\Rightarrow$