MATH 142A: Introduction to Analysis

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Today: Basic properties of the derivative

> Q&A: February 25

Next: Ross § 29

Week 8:

- Homework 7 (due Sunday, February 27)

Differentiability and derivative

Def Let $f: I \to \mathbb{R}$, I open interval. Let $a \in I$.

We say that f is differentiable at $a \in I$, or that f has a derivative at a, if the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} =: f'(a)$$

exists and is finite. If f is differentiabe $\forall a \in I$, we get a function $I \ni a \mapsto f'(a)$ (usually use letter $x \mapsto f'(x)$)

Examples 1) Let $f(x) = x$. Then $\forall a \in \mathbb{R}$ $f'(a) = 1$ (so $f'(x) = 1$)

$\forall a \in \mathbb{R}$

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x - a}{x - a} = \lim_{x \to 0} 1 = 1$$

2) Let $f(x) = \sin x$. Then $f'(x) = \cos x$

$\forall x \in \mathbb{R}$

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right) \cos\left(x + \frac{h}{2}\right)}{h/2} \overset{1E10}{=} 1 \cdot \cos(x)$$

Important examples (limits of functions)

IE 13 $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$

Proof.

① $\frac{\log(1+x)}{x}$ is well-defined on $(-1,+\infty)\setminus\{0\}$

[$b \log a = \log a^b$]

② Write $\frac{\log(1+x)}{x} = \log(1+x)^{\frac{1}{x}}$

③ $\lim_{x\to 0_+} (1+x)^{\frac{1}{x}} = e$: Let (x_n) be a sequence in $(0,1)$

$\lim x_n = 0$. Define $y_n = \frac{1}{x_n}$ Then $\lim y_n = +\infty$

and $\lim_{n\to\infty} (1+x_n)^{\frac{1}{x_n}} = \lim_{n\to\infty} \left(1+\frac{1}{y_n}\right)^{y_n} \overset{\text{IE 12}}{=} e \Rightarrow \lim_{x\to 0^+} (1+x)^{\frac{1}{x}} = e$

④ $\lim_{x\to 0_-} (1+x)^{\frac{1}{x}} = e$. As in ③

⑤ By Thm 20.10 $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$

⑥ log is continuous on $(0,+\infty) \Rightarrow \lim_{x\to 0} \log(1+x)^{\frac{1}{x}} = \log \lim_{x\to 0} (1+x)^{\frac{1}{x}} = \log e = 1$ ∎

Limits of functions. Examples

Warm up

Last time: $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = e$$

① $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{-n} = \lim_{n\to\infty}\left(\frac{n-1}{n}\right)^{-n} =$

② $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{-(n-1)} = \lim_{n\to\infty}\left(1-\frac{1}{n-1}\right)^{-n} =$

③ $\forall\ x<0 \qquad \left(1+\frac{1}{\lceil x\rceil}\right)^{\lceil x\rceil+1} < \left(1+\frac{1}{x}\right)^x < \left(1+\frac{1}{\lceil x\rceil+1}\right)^{\lceil x\rceil}$

④ Fix $\varepsilon>0$. $\exists N\ \forall n>N\ \left|\left(1-\frac{1}{n}\right)^{-(n+1)}-e\right|<\varepsilon,\ \left|\left(1-\frac{1}{n+1}\right)^{-n}-e\right|<\varepsilon$

For $x<$

$$<\left(1+\frac{1}{x}\right)^x - e <$$

Important examples (limits of functions)

IE 14

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1$$

Proof Denote $f(x) := e^x - 1$, so that $x = \log(1+f(x))$

Then $\dfrac{e^x-1}{x} = \dfrac{f(x)}{\log(1+f(x))} = g\circ f(x)$, $x \neq 0$ where

$$g(y) = \begin{cases} \dfrac{y}{\log(1+y)}, & y \in (-1,+\infty)\setminus\{0\} \\ 1, & y = 0 \end{cases}$$

① $f(x)$ is continuous on $\mathbb{R}$, $\lim_{x\to 0} f(x) = 0$, $f(\mathbb{R}) = (-1,+\infty)$

② g is defined on $(-1,+\infty)$, by Thm 20.4, IE 13 g is cont. at 0

$\Rightarrow$ By Thm 20.5 $\lim_{x\to 0} g\circ f(x) = g(f(0)) = g(0) = 1$ ∎

Examples

3) $(e^x)' = e^x$

For any $x \in \mathbb{R}$

$$\lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} e^x \frac{e^h - 1}{h} = e^x \lim_{h\to 0} \frac{e^h - 1}{h} \overset{\text{IE14}}{=} e^x \cdot 1$$

Thm 28.2 f is differentiable at point $a \Rightarrow f$ is continuous at a

Proof. f differentiable at $a \Rightarrow \displaystyle\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a)$

Rewrite $\displaystyle f(x) = f(a) + \frac{f(x)-f(a)}{x-a}(x-a)$

Then $$\lim_{x\to a} f(x) = f(a) + \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \lim_{x\to a} (x-a) = f(a) + f'(a)\cdot 0$$

$$= f(a)$$

$\blacksquare$

Important examples (limits of functions)

IE 15

$\forall \alpha \in \mathbb{R}$ ($\alpha = 0$ is trivial, assume $\alpha \neq 0$)

$$\lim_{x \to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha$$

Proof

1. Write
$$\frac{(1+x)^\alpha - 1}{x} = \underbrace{\frac{e^{\alpha \log(1+x)} - 1}{\alpha \log(1+x)}}_{= g \circ f(x)\text{, } x \neq 0} \cdot \frac{\alpha \log(1+x)}{x}$$

2. Denote $f(x) = \alpha \log(1+x)$
$$g(y) = \begin{cases} \dfrac{e^y - 1}{y}, & y \neq 0 \\ 1, & y = 0 \end{cases}$$

Then by IE 14 g is continuous at 0, so by Thm 20.5 $\lim_{x \to 0} g \circ f(x) = g(0) = 1$.

3. By IE 13 $\lim_{x \to 0} \frac{\alpha \log(1+x)}{x} = \alpha$

■

Derivatives and arithmetic operations

Thm 28.3 Let f and g be differentiable at a, $c \in \mathbb{R}$. Then $c \cdot f$, $f+g$ and $f \cdot g$ are differentiable at a. If additionally $g(a) \neq 0$, then $\frac{f}{g}$ is differentiable at a. Moreover

$$(cf)'(a) = c \cdot f'(a), \quad (f+g)'(a) = f'(a) + g'(a), \quad (f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a)$$

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}$$

Proof. $(cf)'$, $(f+g)'$ – exercise.

$$\lim_{x \to a} \frac{f(x)g(x) - f(a)g(a)}{x-a} = \lim_{x \to a} \frac{\big(f(x) - f(a)\big)g(x) + f(a)\big(g(x) - g(a)\big)}{x-a}$$

$$= f'(a) \cdot g(a) + f(a) \cdot g'(a)$$

If $g(a) \neq 0$, then $\exists\, \delta > 0$ s.t. $(a-\delta, a+\delta)$ $|g(x)| > \frac{|g(a)|}{2} > 0$

$$\lim_{x \to a} \frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x-a} = \lim_{x \to a} \frac{1}{g(x)g(a)} \cdot \frac{f(x)g(a) - f(a)g(x) - f(a)g(a) + f(a)g(a)}{x-a} = \frac{f'(a)g(a) - f(a)g'(a)}{\big(g(a)\big)^2}$$

$\blacksquare$

Derivative of a composition

Thm 28.4 If f is differentiable at a, and g is differentiable at $f(a)$, then $g\circ f$ is differentiable at a and

$$(g\circ f)'(a) = g'(f(a))\cdot f'(a)$$

Remark $$\frac{g(f(x))-g(f(a))}{x-a} = \frac{g(f(x))-g(f(a))}{f(x)-f(a)}\,\frac{f(x)-f(a)}{x-a}$$

Take $f(x)=\begin{cases} x^2\sin\left(\frac{1}{x}\right), & x\neq 0 \\ 0, & x=0\end{cases}$, $g(y)=e^y$: $\lim\limits_{x\to 0}\dfrac{e^{x^2\sin\frac{1}{x}}-e^0}{x^2\sin\left(\frac{1}{x}\right)}$ is not well defined $\left(x_n=\frac{1}{\pi n}\right)$

Proof: ① g is defined on $(f(a)-c, f(a)+c)$ for some $c>0$.

f is cont. at a $\Rightarrow$ $\exists\,\delta>0$ $\forall x$ $(a-\delta, a+\delta)$ $f(x)\in(f(a)-c, f(a)+c)$

$\Rightarrow$ $g\circ f$ is defined on $(a-\delta, a+\delta)$

Need to show that $\lim\limits_{x\to a}\dfrac{g\circ f(x)-g\circ f(a)}{x-a}$ exists (and compute)

Derivative of a composition

Case 1: $\exists\ \eta \leq \delta$ s.t $\forall\ x \in (a-\eta, a+\eta)$ $f(x) \neq f(a)$

Then $\dfrac{g(f(x)) - g(f(a))}{f(x) - f(a)}$ can be written on $(a-\eta, a+\eta)$ as $\varphi \circ f(x)$

where $\varphi(y) = \begin{cases} \dfrac{g(y) - g(f(a))}{y - f(a)}, & y \neq f(a) \\ g'(f(a)), & y = f(a) \end{cases}$ is defined on $(f(a)-c, f(a)+c)$

g is differentiable at $f(a) \Rightarrow \lim\limits_{y \to f(a)} \varphi(y) = \lim\limits_{y \to f(a)} \dfrac{g(y) - g(f(a))}{y - f(a)} = g'(f(a))$ $\overset{\varphi(f(a))}{\shortparallel}$

$\Rightarrow \varphi$ is continuous at $f(a)$. By Thm 20.5 $\lim\limits_{x \to a} \varphi \circ f(x) = \varphi(f(a))$

$$\Rightarrow \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x - a} = \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \cdot \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= g'(f(a)) \cdot f'(a)$$

Case 2: $\exists (x_n),\ \lim x_n = a,\ \forall n\ x_n \neq a,\ f(x_n) = f(a)$

T20.2 $\rightarrow$ f is continuous at a, and $f'(a) = \lim\limits_{n\to\infty} \frac{f(x_n)-f(a)}{x_n - a} = 0$

$\hookrightarrow$ g is continuous at $f(a)$, $f(x_n) = f(a)\ \forall n$, so

if $\lim\limits_{x\to a} \frac{g\circ f(x) - g\circ f(a)}{x-a} = (g\circ f)'(a)$ exists, then

$$(g\circ f)'(a) = \lim_{n\to\infty} \frac{g\circ f(x_n) - g\circ f(a)}{x_n - a} = \lim_{n\to\infty} \frac{g(f(a)) - g(f(a))}{x_n - a} = 0$$

Fix $\varepsilon>0$. g is differentiable at $f(a)$

$$\Rightarrow \exists\, \theta>0\ \forall\, y\in (f(a)-\theta, f(a)+\theta)\setminus\{f(a)\}\ \left|\frac{g(y)-g(f(a))}{y-f(a)}\right| < \overbrace{|g'(f(a))|+1}^{C}$$

Then $\exists\, \delta'>0,\ \delta'<\delta,\ \forall\, x\in(a-\delta', a+\delta')\setminus\{a\}\quad |f(x)-f(a)|<\theta$

Then $\forall x\in (a-\delta', a+\delta')\setminus\{a\}$, $f(x)\neq f(a)$

$$\left|\frac{g(f(x))-g(f(a))}{x-a}\right| = \left|\frac{g(f(x))-g(f(a))}{f(x)-f(a)}\right|\left|\frac{f(x)-f(a)}{x-a}\right| \leq C\cdot\left|\frac{f(x)-f(a)}{x-a}\right|$$

$x\in(a-\delta', a+\delta')$
$f(x) = f(a)$ $\Big|\Rightarrow$ $0 \leq C\cdot 0$

T.20.14
$$\Rightarrow \lim_{x\to a}\left|\frac{g(f(x))-g(f(a))}{x-a}\right| = 0$$ ■