# MATH 142A: Introduction to Analysis

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Today: Derivative of the inverse.
L'Hôpital's rule
> Q&A: March 2

Next: Ross § 31

- Homework 8 (due Sunday, March 6)
- CAPE at www.cape.ucsd.edu

# Derivative of the inverse

$f: I \to J$, $f^{-1}: J \to I$, $\forall x \in I \quad f^{-1} \circ f(x) = x$, $\forall y \in J \quad f \circ f^{-1}(y) = y$

If $f \in D(I)$, $f^{-1} \in D(J)$, then differentiating both sides gives

$$\forall x \in I \quad (f^{-1} \circ f)'(x) = 1 \quad , \quad \forall y \in J \quad (f \circ f^{-1})'(y) = 1$$

By the chain rule

$$(f \circ f^{-1})'(y) = f'(f^{-1}(y)) \cdot (f^{-1})'(y) = 1$$

$$\Rightarrow (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} \qquad (*)$$

If $f^{-1}$ exists and $f$ and $f^{-1}$ are differentiable, then $(f^{-1})'$ is given by $(*)$.

Suppose $f: I \to J$, $f^{-1}: J \to I$ exists and $f$ is differentiable at $x_0 \in I$. Does this imply that $f^{-1}$ is differentiable at $y_0 = f(x_0)$?

## Derivative of the inverse

**Thm. 29.9.** Let $f: I \to J$ be one-to-one and continuous on $I$.

(i) $f$ is differentiable at $x_0$
(ii) $f'(x_0) \neq 0$
$\Rightarrow$ $f^{-1}$ is differetiable at $y_0 = f(x_0)$ and $(f^{-1})'(y_0) = \dfrac{1}{f'(x_0)}$

**Proof.** Need to show that $\displaystyle\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} \in \mathbb{R}$. Fix $\varepsilon > 0$.

① $f'(x_0) \neq 0 \Leftrightarrow \displaystyle\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \neq 0 \Rightarrow \exists\, \delta' \;\forall x \in (x_0 - \delta', x_0 + \delta') \setminus \{x_0\}\; f(x) \neq f(x_0)$

$$\lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} \overset{T20.4}{=} \frac{1}{f'(x_0)} \Rightarrow \exists\, \delta \;\forall x \in (x_0 - \delta, x_0 + \delta) \; \left| \frac{x - x_0}{f(x) - f(x_0)} - \frac{1}{f'(x_0)} \right| < \varepsilon$$

Consider $g := f^{-1}$, $g: J \to I$.

② Thms 18.6, 18.4 $\Rightarrow g \in C(J) \Rightarrow \exists\, \eta > 0 \;\forall y \in (y_0 - \eta, y_0 + \eta)\; |g(y) - \underbrace{g(y_0)}_{= x_0}| < \delta$

③ $\forall y \in (y_0 - \eta, y_0 + \eta) \setminus \{y_0\}$ $\displaystyle\left| \frac{g(y) - g(y_0)}{f(g(y)) - f(g(y_0))} - \frac{1}{f'(x_0)} \right| = \left| \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} - \frac{1}{f'(x_0)} \right| < \varepsilon$

∎

## Examples

1. $\arcsin = \sin^{-1}$, $\arcsin \in D((-1,1))$, $(\arcsin(y))' = \dfrac{1}{\sqrt{1-y^2}}$

$\sin : (-\frac{\pi}{2}, \frac{\pi}{2}) \to (-1,1)$ is a bijection (strictly increasing)

$\forall x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ $\quad \sin'(x) = \cos(x) \neq 0$

Let $y \in (-1,1)$ and let $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ s.t. $\sin x = y$

by Thm 29.9 arcsin is diffentiable at $y$ and

$$\arcsin'(y) = \frac{1}{(\sin(x))'} = \frac{1}{\cos x} = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\sqrt{1-y^2}}$$

2. $\log : (0,+\infty) \to \mathbb{R}$ is the inverse of $x \mapsto e^x$

$e^x \in D(\mathbb{R})$, $(e^x)' = e^x$, $e^x > 0$

$\Rightarrow \forall y \in (0,+\infty)$ log is differentiable at $y$

and $\quad (\log y)' = \dfrac{1}{e^x} \overset{y=e^x}{=} \dfrac{1}{y}$

# L'Hôpital's rule

Consider the limit $\displaystyle\lim_{S \ni x \to a} \frac{f(x)}{g(x)}$ , $a \in \mathbb{R} \cup \{+\infty, -\infty\}$ , $S \subset \mathbb{R}$

- if $\displaystyle\lim_{S \ni x \to a} f(x) =: F \in \mathbb{R}$ , $\displaystyle\lim_{S \ni x \to a} g(x) =: G \in \mathbb{R} \setminus \{0\}$ , then

$$\lim_{S \ni x \to a} \frac{f(x)}{g(x)} \overset{T.20.4}{=} \frac{F}{G}$$

- if $F \in \{+\infty, -\infty\}$ and $G \in \{+\infty, -\infty\}$ $\quad \frac{\infty}{\infty}$ | usual tools don't work

  $F = 0$ and $G = 0$ $\quad \frac{0}{0}$

$f, g$ differentiable $\Rightarrow$ try L'Hôpital's rule

# Generalized mean value theorem (Cauchy's Thm)

**Thm 30.1** $\left.\begin{array}{l} f, g \in C([a,b]) \\ f, g \in D((a,b)) \end{array}\right| \Rightarrow$ $\exists\, x \in (a,b)$ s.t.

$$(f(b)-f(a))\,g'(x) - (g(b)-g(a))\,f'(x) = 0$$

$g(x) = x \to$ Lagr.

**Proof** Consider $h(x) := (f(b)-f(a))\,g(x) - (g(b)-g(a))\,f(x)$

$h \in C([a,b])$

$h \in D((a,b))$

$h(a) = f(b)g(a) - g(b)f(a)$

$h(b) = -f(a)g(b) + f(b)g(a) = h(a)$

Rolle's Thm

$\Longrightarrow$ $\exists\, x \in (a,b)$ s.t. $h'(x) = 0$

$$(f(b)-f(a))\,g'(x) - (g(b)-g(a))\,f'(x) = 0$$

$\blacksquare$

If $g(b) \neq g(a)$, $g'(x) \neq 0$, then

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x)}{g'(x)}$$

# L'Hôpital's Rule

**Thm 30.2** Let $a \in \mathbb{R}$ and $s$ signify $a, a^+, a^-, +\infty$ or $-\infty$.

Suppose that $f$ and $g$ are differentiable (on appropriately chosen intervals) and that $\lim_{x \to s} \frac{f'(x)}{g'(x)} = L$ exists.

Then if

(i) $\lim_{x \to s} f(x) = \lim_{x \to s} g(x) = 0$

OR

(ii) $\lim_{x \to s} |g(x)| = \infty$

$$\Rightarrow \lim_{x \to s} \frac{f(x)}{g(x)} = \lim_{x \to s} \frac{f'(x)}{g'(x)} = L$$

**Proof** Only for $s = a^-$ and for $s = +\infty$ (other cases: exercise)

# Proof of L'Hôpital's rule

① Suppose $-\infty < L \leq +\infty$. Take $L_1 < L$.

$\lim_{x \to s} \frac{f'(x)}{g'(x)}$ exists $\Rightarrow \exists\, c < s$ s.t. $f, g \in D((c,s))$, $\forall x \in (c,s)\ g'(x) \neq 0$

By Darboux's thm. either $\forall x \in (c,s)\ g'(x) > 0$ or $\forall x \in (c,s)\ g'(x) < 0$

Cor 29.7

$\Rightarrow \quad \{x \in (c,s) : g(x)\}$ has at most one point

$\Rightarrow \exists\, c' \in (c,s)\ \forall x \in (c',s)\quad g(x) \neq 0$

Take $K \in (L_1, L)$. $\lim_{x \to s} \frac{f'(x)}{g'(x)} = L > K \Rightarrow \exists\, \alpha > c'\ \forall x \in (\alpha, s)\ \frac{f'(x)}{g'(x)} > K$

By Cauchy's thm $\forall [x,y] \subset (\alpha, s)\ \exists\, z \in (x,y)$ s.t.

$$(f(y) - f(x))\, g'(z) = (g(y) - g(x))\, f'(z) \Rightarrow \frac{f(y) - f(x)}{g(y) - g(x)} = \frac{f'(z)}{g'(z)} > K$$

If (i) holds, take $\lim_{y \to s} \frac{f(y) - f(x)}{g(y) - g(x)} = \frac{f(x)}{g(x)} \geq K > L_1 \quad \forall x \in (\alpha, s)$

# Proof of L'Hôpital's rule

If (ii) holds, then $\exists\, \alpha_1 \in (\alpha, s)$ s.t. $\forall\, [x,y] \subset (\alpha_1, s)$ $\quad \dfrac{g(y)-g(x)}{g(y)} > 0$

$$\Rightarrow \forall\, [x,y] \subset (\alpha_1, s) \qquad \frac{f(y)-f(x)}{g(y)-g(x)} \cdot \frac{g(y)-g(x)}{g(y)} > K \cdot \frac{g(y)-g(x)}{g(y)}$$

$$\Rightarrow \frac{f(y)}{g(y)} = \frac{f(x)}{g(y)} + \frac{f(y)-f(x)}{g(y)} > \frac{f(x)}{g(y)} + K \cdot \frac{g(y)-g(x)}{g(y)} = K + \frac{f(x)-Kg(x)}{g(y)}$$

Take the limit (for any fixed $x \in (\alpha_1, s)$)

$$\lim_{y \to s} \frac{f(x)-Kg(x)}{g(y)} = 0 \Rightarrow \exists\, \alpha_2 \in (\alpha_1, s) \text{ s.t. } \forall\ y \in (\alpha_2, s)$$

$$\frac{f(x)-Kg(x)}{g(y)} > \frac{L_1-K}{2} \Rightarrow \frac{f(y)}{g(y)} > K + \frac{L_1-K}{2} = \frac{K+L_1}{2} > L_1$$

Conclusion: $\forall\, L_1 < L \quad \exists\, \alpha_2 < s \quad \forall x \in (\alpha_2, s) \quad \dfrac{f(x)}{g(x)} > L_1 \qquad \text{(A)}$

# Proof of L'Hôpital's rule

② If $-\infty \le L < +\infty$, then

$$\forall L_2 > L \quad \exists \beta_2 < s \quad \forall x \in (\beta_2, s) \quad \frac{f(x)}{g(x)} < L_2 \qquad (B)$$

③ Suppose $L \in \mathbb{R}$. Fix $\varepsilon > 0$. Take $L_1 = L - \varepsilon$, $L_2 = L + \varepsilon$

$$(A) \Rightarrow \exists \alpha_2 < s \quad \forall x \in (\alpha_2, s) \quad \frac{f(x)}{g(x)} - L > L_1 - L = -\varepsilon$$

$$(B) \Rightarrow \exists \beta_2 < s \quad \forall x \in (\beta_2, s) \quad \frac{f(x)}{g(x)} - L < L_2 - L = \varepsilon$$

$$\Rightarrow \forall x \in (\max\{\alpha_2, \beta_2\}, s) \quad \left|\frac{f(x)}{g(x)} - L\right| < \varepsilon \Rightarrow \lim_{x \to s} \frac{f(x)}{g(x)} = L$$

Suppose $L = +\infty$. Fix $M > 0$. Take $L_1 = M$.

$$(A) \Rightarrow \exists \alpha_2 < s \quad \forall x \in (\alpha_2, s) \quad \frac{f(x)}{g(x)} > M \Rightarrow \lim_{x \to s} \frac{f(x)}{g(x)} = +\infty = L$$

Suppose $L = -\infty$. Fix $M > 0$. Take $L_2 = -M \Rightarrow \lim_{x \to s} \frac{f(x)}{g(x)} = -\infty$ ∎

# Examples

1. For any $\alpha > 0$

$$\lim_{x \to +\infty} \frac{\log x}{x^\alpha} = \lim_{x \to +\infty} \frac{\frac{1}{x}}{\alpha x^{\alpha-1}} = \lim_{x \to +\infty} \frac{1}{\alpha x^\alpha} = 0$$

2. $\forall$ $a > 1$ and $0 < \alpha < n$

$$\lim_{x \to +\infty} \frac{x^\alpha}{a^x} = \lim_{x \to +\infty} \frac{\alpha x^{\alpha-1}}{\log a \cdot a^x} = \lim_{x \to +\infty} \frac{\alpha(\alpha-1) x^{\alpha-2}}{(\log a)^2 a^x} =$$

$$= \cdots = \lim_{x \to +\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1) x^{\alpha-n}}{(\log a)^n a^x} = 0$$

3. $$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$$

# Examples

3. $f: \mathbb{R} \to (0,+\infty)$ , $f(x) = a^x$ $(a>0 , a \neq 1)$

$$f(x) = e^{\log a^x} = e^{x \cdot \log a} \Rightarrow \forall x \in \mathbb{R} \quad f'(x) \overset{T28.4}{=} e^{x \cdot \log a} \cdot \log a = a^x \cdot \log a$$

4. $\log_a : (0,+\infty) \to \mathbb{R}$ is the inverse of $x \mapsto a^x$ , $\forall x \in \mathbb{R}$ $a^x > 0$,

so $\log_a \in D((0,+\infty))$ and

$$(\log_a y)' \overset{T29.9}{=} \frac{1}{\log a \cdot a^x} \overset{a^x = y}{=} \frac{1}{\log a \cdot y}$$