MATH 142A: Introduction to Analysis

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Today: Taylor's formula

Little-o/big-O notation

> Q&A: March 7

Next: -

Week 10:

- Homework 9 (due Sunday, March 13)
- CAPE at www.cape.ucsd.edu

Taylor's formula

Let $f: I \to \mathbb{R}$, f has derivatives up to order n at $x_0 \in I$.

Taylors formula :

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x_0;x)$$

Taylor's Thm: If $f \in D^{(n)}(\bar{I})$, $f \in D^{(n+1)}(I)$, $f, f', f^{(2)}, \ldots, f^{(n)} \in C(\bar{I})$.

then for any function $\varphi \in C(\bar{I})$, $\varphi \in D(I)$, $\forall x \in I$ $\varphi'(x) \neq 0$

there exists $\xi \in I$ s.t.

$$R_n(x_0;x) = \frac{\varphi(x) - \varphi(x_0)}{\varphi'(\xi)\, n!} f^{(n+1)}(\xi)(x-\xi)^n$$

Cauchy's form of the remainder term $R_n(x_0;x) = \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n (x-x_0)$

Lagrange's form of the remainder term $R_n(x_0;x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$

Example

IE 19 Let $f(x) = (1+x)^\alpha$, $\alpha \in \mathbb{R}$, $x > -1$. Then (Lecture 22)

$\forall n \in \mathbb{N}$ $\quad f^{(n)}(x) = \alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n}$

Taylor's formula at $x_0 = 0$:

$$(1+x)^\alpha = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + R_n(0;x)$$

Cauchy's form of the remainder (ξ between x and 0)

$$R_n(0;x) = \frac{\alpha(\alpha-1)\cdots(\alpha-n)\cdot(1+\xi)^{\alpha-n-1}}{n!}\cdot(x-\xi)^n x = \frac{\alpha(\alpha-1)\cdots(\alpha-n)(1+\xi)^{\alpha-1}}{n!}\left(\frac{x-\xi}{1+\xi}\right)^n x$$

For $|x|<1$ $\quad \left|\frac{x-\xi}{1+\xi}\right| = \frac{|x|-|\xi|}{|1+\xi|} \leq \frac{|x|-|\xi|}{1-|\xi|} = 1 - \frac{1-|x|}{1-|\xi|} \leq 1 - \frac{1-|x|}{1} = |x|$, so

$$|R_n(0;x)| \leq (1+|x|)^{\alpha-1}\frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!}|x|^{n+1} =: C_n \; ; \quad \frac{C_{n+1}}{C_n} = \left|\frac{n+1}{\alpha-(n+1)}\right||x| \to |x| \text{ as } n\to\infty$$

$$|x| < 1$$

$\Rightarrow \exists\, q \in (x,1)\ \exists\, N \in \mathbb{N}\ \forall k \in \mathbb{N}\ \ \frac{C_{N+k}}{C_N} < q^k \Rightarrow \lim_{n\to\infty} C_n = 0 \Rightarrow \lim_{n\to\infty} R_n(0;x) = 0$

$\alpha = n \in \mathbb{N} \Rightarrow$ Newton binomial Thm ; if $\alpha = -1 \Rightarrow$ geometric series

Taylor series. Analytic functions

Def 31.18. If the function $f(x)$ has derivatives of all orders $n \in \mathbb{N}$ at x_0, we call the series

$$f(x_0) + \frac{1}{1!} f'(x_0)(x-x_0) + \frac{1}{2!} f''(x_0)(x-x_0)^2 + \frac{1}{n!} f^{(n)}(x_0)(x-x_0)^n + \cdots$$

the Taylor series of f at point x_0.

Remarks 1) If f has derivatives of all orders at x_0, this does not imply that the Taylor series of f at x_0 converges

2) If the Taylor series of f at x_0 converges, than this does not imply that $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = f(x)$ $(*)$

Functions that satisfy $(*)$ are called analytic

Example of a non-analytic function $f(x) = \begin{cases} 0, & x=0 \\ e^{-\frac{1}{x^2}}, & x \neq 0 \end{cases}$

$f^{(n)}(0) = 0 \quad \forall\, n = 0, 1, 2, \cdots$ (exercise)

Comparison of the Asymptotic Behavior of functions

Def 31.19 • Let $a \in \mathbb{R}$ and $s \in \{a^-, +\infty\}$. For $f, g : (c, s) \to \mathbb{R}$, $c < s$, we say that f is infinitesimal compared with g as x tends to s, and write $f = o(g)$ as $x \to s$ if there exist $c' \geq c$ and $h : (c', s) \to \mathbb{R}$ such that

$$f(x) = g(x) \cdot h(x) \quad \text{on} \quad (c', s) \quad \text{and} \quad \lim_{x \to s} h(x) = 0$$

• Let $a \in \mathbb{R}$ and $s \in \{a^+, -\infty\}$. For $f, g : (s, c) \to \mathbb{R}$, $c > s$ we say that f is infinitesimal compared with g as x tends to s, and write $f = o(g)$ as $x \to s$, if there exist $c' \leq c$ and $h : (s, c') \to \mathbb{R}$ such that

$$f(x) = g(x) \cdot h(x) \quad \text{on} \quad (s, c') \quad \text{and} \quad \lim_{x \to s} h(x) = 0$$

• $f = o(g)$ as $x \to a$ if $f = o(g)$ as $x \to a^+$ and $f = o(g)$ as $x \to a^-$

Examples

1) $\overset{f}{x^2} = \overset{g}{x} \cdot \overset{h}{x} \Rightarrow x^2 = o(x)$ as $x \to 0$

2) $\overset{f}{x} = \overset{h}{\frac{1}{x}} \cdot \overset{g}{x^2}$ on $(0, +\infty) \Rightarrow x = o(x^2)$ as $x \to +\infty$

3) $\frac{1}{x^2} = \frac{1}{x} \cdot \frac{1}{x}$ on $(0, +\infty) \Rightarrow \frac{1}{x^2} = o\left(\frac{1}{x}\right)$ as $x \to +\infty$

4) $\frac{1}{x} = x \cdot \frac{1}{x^2}$ on $(0, 1) \Rightarrow \frac{1}{x} = o\left(\frac{1}{x^2}\right)$ as $x \to 0^+$

5) For $a > 1$, $\lim\limits_{x \to +\infty} \frac{x^n}{a^x} = 0$, $x^n = a^x \cdot \frac{x^n}{a^x}$ on $(0, +\infty) \Rightarrow x^n = o(a^x)$ as $x \to +\infty$

6) $\forall\, a > 0, a \neq 1, \forall\, \alpha > 0$ $\lim\limits_{x \to +\infty} \frac{\log_a x}{x^\alpha} = 0 \Rightarrow \log_a x = o(x^\alpha)$ as $x \to +\infty$

7) $x = x \cdot 1 \Rightarrow x = o(1)$ as $x \to 0$

8) $\left(\frac{1}{x} + \sin x\right) \cdot x = O(x)$ as $x \to \infty$

9) $(2 + \sin x) \cdot x \asymp x$ as $x \to \infty$, but $(1 + \sin x)x$ is not of the same order as x as $x \to \infty$

10) $x^2 + x = x^2\left(1 + \frac{1}{x}\right) \Rightarrow x^2 + x \sim x^2$ as $x \to \infty$

Comparison of the Asymptotic Behavior of functions

Def 31.19
- Let $a \in \mathbb{R}$ and $s \in \{a^-, +\infty\}$. For $f, g : (c, s) \to \mathbb{R}$, $c < s$, we write $f = O(g)$ as $x \to s$ if there exist $c' \geq c$ and $B : (c', s) \to \mathbb{R}$ such that $f(x) = g(x) B(x)$ on (c', s) and B is bounded on (c', s)
- Let $a \in \mathbb{R}$ and $s \in \{a^+, -\infty\}$. For $f, g : (s, c) \to \mathbb{R}$, $c > s$ we write $f = O(g)$ as $x \to s$, if there exist $c' \leq c$, $B : (s, c') \to \mathbb{R}$ s.t. $f(x) = g(x) \cdot B(x)$ on (s, c') and B is bounded on (c', s)
- $f = O(g)$ as $x \to a$ if $f = O(g)$ as $x \to a^+$ and $f = O(g)$ as $x \to a^-$
- We say that f and g are of the same order as $x \to s$ and write $f \asymp g$ as $x \to s$ if $f = O(g)$ and $g = O(f)$ as $x \to s$

$\Leftrightarrow$ $\exists$ $c_1, c_2 \in (0, +\infty)$ s.t. $c_1 |g(x)| \leq |f(x)| \leq c_2 |g(x)|$ on the corresponding interval

Comparison of the Asymptotic Behavior of functions

Def 31.19
- Let $a \in \mathbb{R}$ and $s \in \{a^-, +\infty\}$. For $f, g : (c, s) \to \mathbb{R}$, $c < s$, we say that f is equivalent to g as x tends to s, and write $f \sim g$ as $x \to s$, if there exist $c' \geq c$ and $\gamma : (c', s) \to \mathbb{R}$ such that

$$f(x) = g(x) \cdot \gamma(x) \quad \text{on} \quad (c', s) \quad \text{and} \quad \lim_{x \to s} \gamma(x) = 1$$

- Let $a \in \mathbb{R}$ and $s \in \{a^+, -\infty\}$. For $f, g : (s, c) \to \mathbb{R}$, $c > s$ we say that f is equivalent to g as x tends to s, and write $f \sim g$ as $x \to s$, if there exist $c' \leq c$ and $\gamma : (s, c') \to \mathbb{R}$ such that

$$f(x) = g(x) \cdot \gamma(x) \quad \text{on} \quad (s, c') \quad \text{and} \quad \lim_{x \to s} \gamma(x) = 1$$

- $f \sim g$ as $x \to a$ if $f \sim g$ as $x \to a^+$ and $f \sim g$ as $x \to a^-$

Taylor's formula

Lemma 31.20 Let $x_0 \in \mathbb{R}$, $\bar{I}$ be a closed interval with endpoint x_0, let φ be a function defined on $\bar{I}$, $\varphi \in D^{(n)}(\bar{I})$, and

$$\varphi(x_0) = \varphi'(x_0) = \dots = \varphi^{(n)}(x_0) = 0. \quad \text{Then}$$

$$\varphi(x) = o((x-x_0)^n) \quad \text{as } x \to x_0 \text{ along } \bar{I}. \qquad (**)$$

Proof. (By induction). If $n=1$, then

$$\varphi(x) = \varphi(x_0) + \frac{\varphi(x)-\varphi(x_0)}{x-x_0}(x-x_0), \ \varphi(x_0)=0, \ \varphi'(x_0)=0 \Rightarrow \varphi(x) = o(x-x_0) \text{ as } x \to x_0 \text{ along } \bar{I}$$

Suppose $(**)$ holds for $n=k-1$. Consider $\varphi' \in D^{(k-1)}(\bar{I})$, $\varphi'(x_0)=0$

$$(\varphi')'(x_0) = (\varphi')''(x_0) = \dots = (\varphi')^{(k-1)}(x_0) = 0 \Rightarrow \varphi'(x) = o((x-x_0)^{k-1}) \quad \text{as } x \to x_0 \text{ along } \bar{I}$$

By Lagrange's thm, for $x \in \bar{I}$ close enough to x_0 $\exists \, \xi$ between x_0 and x

$$\varphi(x) - \varphi(x_0) = \varphi'(\xi)(x-x_0) = h(\xi)(\xi-x_0)^{k-1}(x-x_0), \quad h(x) \to 0 \quad \text{as } \bar{I} \ni x \to x_0$$

$$\Rightarrow |\varphi(x)| \le |h(\xi)||x-x_0|^k \Rightarrow \varphi(x) = o((x-x_0)^k),$$ proves induction step $\blacksquare$

Taylor's formula (local). Peano's form of the remainder

Thm 31.21 Let $x_0 \in \mathbb{R}$, $\bar{I}$ be a closed interval with endpoint x_0, let f be a function defined on $\bar{I}$, $f \in D^{(n)}(\bar{I})$. Then

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$+\, o\left((x-x_0)^n\right) \quad \text{as } x \to x_0,\ x \in \bar{I}$$

Proof. Apply Lemma 31.20 with $\varphi(x) = R_n(x_0; x)$ $\blacksquare$

Remark If $f \in D^{(n+1)}(I)$ and $f^{(n+1)}$ is bounded near x_0, then

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$+\, O\left((x-x_0)^{n+1}\right) \quad \text{as } x \to x_0,\ x \in \bar{I}$$

Examples

1) Asymptotic formulas as $x \to 0$

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{x + o(x^2)}{x}$$

$$= 1 + \lim_{x\to 0} \frac{o(x^2)}{x^2} \cdot x = 1$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + o(x^{n+1})$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + o(x^{2n+2})$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + o(x^{2n+1})$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n+1} x^n}{n} + o(x^n)$$

$$(1+x)^\alpha = 1 + \frac{\alpha}{1!} x + \frac{\alpha(\alpha-1)}{2!} x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n + o(x^n)$$

2) Approximate sin by a polynomial P_n s.t. $\max_{x\in[-1,1]} |\sin x - P_n(x)| \le 10^{-3}$

Take $P_n = P_n(0;x)$ Taylors polynomial at 0. By Lagrange's form

$$|R_{2n+2}(0;x)| = \left|\frac{\sin(\xi + \frac{\pi}{2}(2n+3))}{(2n+3)!}\right| |x|^{2n+3} \le \frac{1}{(2n+3)!} < \frac{1}{1000} \text{ for } n=2$$

$$\Rightarrow \sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} \text{ on } [-1,1]$$