# MATH 142A: Introduction to Analysis

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Today: Taylor's formula
Little-o/big-O notation
> Q&A: March 7

Next: -

Week 10:

- Homework 9 (due Sunday, March 13)
- CAPE at www.cape.ucsd.edu

# Taylor's formula

Let $f: I \to \mathbb{R}$, $f$ has derivatives up to order $n$ at $x_0 \in I$.

Taylors formula :

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x_0;x)$$

Taylor's Thm: If $f \in D^{(n)}(\bar{I})$, $f \in D^{(n+1)}(I)$, $f, f', f^{(2)}, \dots, f^{(n)} \in C(\bar{I})$.

then for any function $\varphi \in C(\bar{I})$, $\varphi \in D(I)$, $\forall x \in I$ $\varphi'(x) \neq 0$

there exists $\xi \in I$ s.t.

$$R_n(x_0;x) = \frac{\varphi(x) - \varphi(x_0)}{\varphi'(\xi)\, n!} f^{(n+1)}(\xi)(x-\xi)^n$$

Cauchy's form of the remainder term $R_n(x_0;x) = \dfrac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n (x-x_0)$

Lagrange's form of the remainder term $R_n(x_0;x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$

## Example

**IE 19** Let $f(x) =$ , $\alpha \in \mathbb{R}$, $x > -1$. Then (Lecture 22)

$\forall n \in \mathbb{N}$ $f^{(n)}(x) =$

Taylor's formula at $x_0 = 0$:

$$(1+x)^\alpha =$$

Cauchy's form of the remainder ($\xi$ between $x$ and 0)

$R_n(0;x) =$

For $|x|<1$ $\left|\frac{x-\xi}{1+\xi}\right| =$ , so

$$|R_n(0;x)| \leq (1+|x|)^{\alpha-1} \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!} |x|^{n+1} =: C_n ;$$

$\alpha = n \in \mathbb{N} \Rightarrow$ Newton binomial Thm ; if $\alpha = -1 \Rightarrow$ geometric series

# Taylor series. Analytic functions

**Def 31.18.** If the function $f(x)$ has derivatives of all orders $n \in \mathbb{N}$ at $x_0$, we call the series

$$f(x_0) + \frac{1}{1!} f'(x_0)(x-x_0) + \frac{1}{2!} f''(x_0)(x-x_0)^2 + \frac{1}{n!} f^{(n)}(x_0)(x-x_0)^n + \cdots$$

the Taylor series of $f$ at point $x_0$.

**Remarks** 1) If $f$ has derivatives of all orders at $x_0$, this does not imply that the Taylor series of $f$ at $x_0$ converges

2) If the Taylor series of $f$ at $x_0$ converges, than this does not imply that $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = f(x)$ $(*)$

Functions that satisfy $(*)$ are called analytic

Example of a non-analytic function $f(x) = \begin{cases} 0, & x=0 \\ e^{-\frac{1}{x^2}}, & x \neq 0 \end{cases}$

$f^{(n)}(0) = 0 \quad \forall\, n = 0, 1, 2, \cdots$ (exercise)

# Comparison of the Asymptotic Behavior of functions

**Def 31.19**

- Let $a \in \mathbb{R}$ and $s \in \{a^-, +\infty\}$. For $f, g : (c, s) \to \mathbb{R}$, $c < s$, we say that $f$ is infinitesimal compared with $g$ as $x$ tends to $s$, and write

  if there exist $c' \geq c$ and $h : (c', s) \to \mathbb{R}$ such that

  on $(c', s)$ and

- Let $a \in \mathbb{R}$ and $s \in \{a^+, -\infty\}$. For $f, g : (s, c) \to \mathbb{R}$, $c > s$ we say that $f$ is infinitesimal compared with $g$ as $x$ tends to $s$, and write $f = o(g)$ as $x \to s$, if there exist $c' \leq c$ and $h : (s, c') \to \mathbb{R}$ such that

  $$f(x) = g(x) \cdot h(x) \text{ on } (s, c') \text{ and } \lim_{x \to s} h(x) = 0$$

- $f = o(g)$ as $x \to a$ if $f = o(g)$ as $x \to a^+$ and $f = o(g)$ as $x \to a^-$

# Examples

1) $x^2 = x \cdot x \quad \Rightarrow$ as $x \to 0$

2) $x = \frac{1}{x} \cdot x^2$ on $(0, +\infty) \quad \Rightarrow$ as $x \to +\infty$

3) $\frac{1}{x^2} = \frac{1}{x} \cdot \frac{1}{x}$ on $(0, +\infty) \quad \Rightarrow$ as $x \to +\infty$

4) $\frac{1}{x} = x \cdot \frac{1}{x^2}$ on $(0, 1) \quad \Rightarrow$ as $x \to 0^+$

5) For $a > 1$, $\lim\limits_{x \to +\infty} \frac{x^n}{a^x} = 0$, $x^n = a^x \cdot \frac{x^n}{a^x}$ on $(0, +\infty) \quad \Rightarrow$ as $x \to +\infty$

6) $\forall\, a > 0, a \neq 1, \ \forall\, \alpha > 0 \quad \lim\limits_{x \to +\infty} \frac{\log_a x}{x^\alpha} = 0 \quad \Rightarrow$ as $x \to +\infty$

7) $x = x \cdot 1 \quad \Rightarrow$ as $x \to 0$

8) $\left(\frac{1}{x} + \sin x\right) \cdot x =$ as $x \to \infty$

9) $(2 + \sin x) \cdot x \asymp x$ as $x \to \infty$, but $(1 + \sin x)x$ is not of the same order as $x$ as $x \to \infty$

10) $x^2 + x = x^2\left(1 + \frac{1}{x}\right) \Rightarrow$

# Comparison of the Asymptotic Behavior of functions

**Def 31.19**

- Let $a \in \mathbb{R}$ and $s \in \{a^-, +\infty\}$. For $f, g : (c, s) \to \mathbb{R}$, $c < s$, we write

  if there exist $c' \geq c$ and $B: (c', s) \to \mathbb{R}$ such that

  on $(c', s)$ and

- Let $a \in \mathbb{R}$ and $s \in \{a^+, -\infty\}$. For $f, g : (s, c) \to \mathbb{R}, c > s$ we write $f = O(g)$ as $x \to s$, if there exist $c' \leq c$, $B : (s, c') \to \mathbb{R}$ s.t. $f(x) = g(x) \cdot B(x)$ on $(s, c')$ and $B$ is bounded on $(c', s)$

- $f = O(g)$ as $x \to a$ if $f = O(g)$ as $x \to a^+$ and $f = O(g)$ as $x \to a^-$

- We say that $f$ and $g$ are of the same order as $x \to s$ and write $f \asymp g$ as $x \to s$ if $f = O(g)$ and $g = O(f)$ as $x \to s$

  $\Leftrightarrow \exists \ c_1, c_2 \in (0, +\infty)$ s.t. $c_1 |g(x)| \leq |f(x)| \leq c_2 |g(x)|$ on the corresponding interval

# Comparison of the Asymptotic Behavior of functions

**Def 31.19**

- Let $a \in \mathbb{R}$ and $s \in \{a^-, +\infty\}$. For $f, g : (c, s) \to \mathbb{R}$, $c < s$, we say that $f$ is equivalent to $g$ as $x$ tends to $s$, and write as $x \to s$, if there exist $c' \geq c$ and $\gamma : (c', s) \to \mathbb{R}$ such that

  on $(c', s)$ and

- Let $a \in \mathbb{R}$ and $s \in \{a^+, -\infty\}$. For $f, g : (s, c) \to \mathbb{R}$, $c > s$ we say that $f$ is equivalent to $g$ as $x$ tends to $s$, and write $f \sim g$ as $x \to s$, if there exist $c' \leq c$ and $\gamma : (s, c') \to \mathbb{R}$ such that

$$f(x) = g(x) \cdot \gamma(x) \text{ on } (s, c') \text{ and } \lim_{x \to s} \gamma(x) = 1$$

- $f \sim g$ as $x \to a$ if $f \sim g$ as $x \to a^+$ and $f \sim g$ as $x \to a^-$

# Taylor's formula

**Lemma 31.20** Let $x_0 \in \mathbb{R}$, $\bar{I}$ be a closed interval with endpoint $x_0$, let $\varphi$ be a function defined on $\bar{I}$, $\varphi \in D^{(n)}(\bar{I})$, and

$$\varphi(x_0)=\varphi'(x_0)=\dots=\varphi^{(n)}(x_0)=0.$$

Then

as $x \to x_0$ along $\bar{I}$.

**Proof.** (By induction). If $n=1$, then

$\varphi(x) =$

Suppose $(**)$ holds for $n=k-1$. Consider $\varphi'$

as $x \to x_0$ along $\bar{I}$

By Lagrange's thm, for $x \in \bar{I}$ close enough to $x_0$ $\exists\, \xi$ between $x_0$ and $x$

as $\bar{I} \ni x \to x_0$

$\Rightarrow |\varphi(x)|$ $\Rightarrow$ , proves induction step ∎

# Taylor's formula (local). Peano's form of the remainder

**Thm 31.21** Let $x_0 \in \mathbb{R}$, $\bar{I}$ be a closed interval with endpoint $x_0$, let $f$ be a function defined on $\bar{I}$, $f \in D^{(n)}(\bar{I})$. Then

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$, \quad x \in \bar{I}$$

**Proof.** Apply Lemma 31.20 with $\varphi(x) =$

**Remark** If $f \in D^{(n+1)}(I)$ and $f^{(n+1)}$ is bounded near $x_0$, then

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$+ O\left((x-x_0)^{n+1}\right) \quad \text{as} \quad x \to x_0, \quad x \in \bar{I}$$

# Examples

1) Asymptotic formulas as $x \to 0$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + O(x^{n+1})$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + O(x^{2n+3})$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + O(x^{2n+2})$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n+1} x^n}{n} + O(x^{n+1})$$

$$(1+x)^\alpha = 1 + \frac{\alpha}{1!} x + \frac{\alpha(\alpha-1)}{2!} x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n + O(x^{n+1})$$

2) Approximate sin by a polynomial $P_n$ s.t. $\max\limits_{x \in [-1,1]} |\sin x - P_n(x)| \leq 10^{-3}$

Take $P_n = P_n(0;x)$ Taylors polynomial at 0. By Lagrange's formula

$$|R_{2n+2}(0;x)| = \left| \frac{\sin(\xi + \frac{\pi}{2}(2n+3))}{(2n+3)!} \right| |x|^{2n+3}$$