MATH 142A: Introduction to Analysis

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Today: Set of real numbers and completeness axiom

> Q&A: January 10

Next: Ross § 7

Week 2:

- homework 1 (due Friday, January 14)

Maximum and minimum

Let $\mathbb{F}$ be an ordered field and let $S \subset \mathbb{F}$, $S \neq \emptyset$

Def $(s_0 = \max S) :\Leftrightarrow (s_0 \in S \wedge \forall s \in S\ (s \leq s_0))$ maximum of S

$(s_0 = \min S) :\Leftrightarrow (s_0 \in S \wedge \forall s \in S\ (s_0 \leq s))$ minimum of S

Examples 1. Any finite nonempty subset of $\mathbb{F}$ has max and min

Take $\mathbb{F} = \mathbb{Q}$ or $\mathbb{R}$ $\max\{-\frac{1}{2}, 0, 1, \frac{7}{3}\} = \frac{7}{3}$, $\min\{-\frac{1}{2}, 0, 1, \frac{7}{3}\} = -\frac{1}{2}$

2. For $\mathbb{F} = \mathbb{R}$ and $a < b$, denote

$[a,b] := \{x \in \mathbb{R} : a \leq x \leq b\}$ $(a,b) := \{x \in \mathbb{R} : a < x < b\}$

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(a) $\max[a,b] = \max(a,b] = b$ $\min[a,b] = \min[a,b) = a$

$$\left(\forall x \in (a,b]\ (x \leq b) \wedge b \in (a,b]\right) \Rightarrow b = \max(a,b]$$

Maximum and minimum

(b) $\max[a,b)$, $\max(a,b)$, $\min(a,b]$, $\min(a,b)$ do not exist

(Proof by contradiction:

Suppose that $\exists x_0 \in \mathbb{R}$ s.t. $x_0 = \max[a,b)$. Then

$x_0 \in [a,b) \wedge \forall x \in [a,b) \, (x \le x_0)$. But

$x_0 \in [a,b) \Rightarrow x_0 < b \Rightarrow x_0 < \frac{x_0+b}{2} < b \Rightarrow \left(\frac{x_0+b}{2} \in [a,b) \wedge \frac{x_0+b}{2} > x_0\right)$

Contradiction)

3. Recall $\max[0,\sqrt{2}] = \max\{x \in \mathbb{R} : 0 \le x \le \sqrt{2}\} = \sqrt{2}$

But $\max\{q \in \mathbb{Q} : 0 \le q \le \sqrt{2}\}$ does not exist

(Suppose $q_0 = \max\{q \in \mathbb{Q} : 0 \le q \le \sqrt{2}\}$. Then $q_0 < \sqrt{2}$, $\sqrt{2} - q_0 > 0$.

$\sqrt{2} - q_0 > 0 \Rightarrow \exists n_0 \in \mathbb{N}$ s.t. $0 < \frac{1}{n_0} < \sqrt{2} - q_0$. Then

$$q_0 < q_0 + \frac{1}{n_0} < \sqrt{2}$$ contradiction

Upper / lower bound

Let $\mathbb{F}$ be an ordered field and let $S \subset \mathbb{F}$, $S \neq \emptyset$

<u>Def</u> If $M \in \mathbb{F} \wedge \forall s \in S \ \ s \leq M$, then M is called an upper bound of S and S is called bounded above

If $m \in \mathbb{F} \wedge \forall s \in S \ \ m \leq s$, then m is called a lower bound of S and S is called bounded below

S is called bounded, if it is bounded above and bounded below

<u>Examples</u> 1. Intervals $[a,b]$, $[a,b)$, $(a,b]$, (a,b) are bounded: any $m \leq a$ is a lower bound, any $M \geq b$ is an upper bound for these sets.

2. If $s_0 = \max S$, then any $M \geq s_0$ is an upper bound for S.

3. Sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are not bounded above.

Supremum and infimum

Let $\mathbb{F}$ be an ordered field and let $S \subset \mathbb{F}$, $S \neq \emptyset$

<u>Def</u> If S is bounded above and S has a least upper bound then we call it the supremum of S, $\sup S$

$$(M = \sup S) := (\forall s \in S\ (s \leq M)) \wedge (\forall M_1 < M\ \exists s_1 \in S\ (s_1 > M_1)) \quad (*)$$

If S is bounded below and S has a greatest lower bound then we call it the infimum of S, $\inf S$

$$(m = \inf S) := (\text{exercise})$$

<u>Examples</u> 1. If $\max S$ exists, then $\sup S = \max S$ (similarly inf)

(take $s_1 = M$ in definition $(*)$)

2. $\sup [a,b] = \sup [a,b) = \sup (a,b] = \sup (a,b) = b$ (similarly for inf)

(for $[a,b)$: if $M_1 < b$, then $\underbrace{\frac{b+M_1}{2}}_{= s_1} \in [a,b) \wedge \frac{b+M_1}{2} > M_1$)

Completeness axiom

3. (a) $\mathbb{F}=\mathbb{R}$ $\quad \max[0,\sqrt{2}] = \max\{x\in\mathbb{R} : 0\le x\le\sqrt{2}\} = \sqrt{2}$

$\sup[0,\sqrt{2}] = \sup\{x\in\mathbb{R} : 0\le x\le\sqrt{2}\} = \sqrt{2}$

(b) $\mathbb{F}=\mathbb{R}$ $\quad \max\{x\in\mathbb{Q} : 0\le x\le\sqrt{2}\}$ does not exist

$\sup\{x\in\mathbb{Q} : 0\le x\le\sqrt{2}\} = \sqrt{2}$

(c) $\mathbb{F}=\mathbb{Q}$ $\quad \max\{x\in\mathbb{Q} : 0\le x\le\sqrt{2}\}$ does not exist

$\sup\{x\in\mathbb{Q} : 0\le x\le\sqrt{2}\}$ does not exist

Completeness Axiom

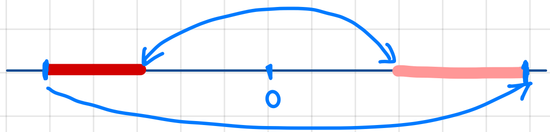
Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound, i.e., $\sup S$ exists and is a real number.

$$(S\neq\emptyset)\wedge(S \text{ bounded above}) \Rightarrow \exists \sup S$$

Satisfied by $\mathbb{R}$ (by definition), not satisfied by $\mathbb{Q}$.

Corollary 4.5

Let $S \subset \mathbb{R}$. $(S \neq \emptyset) \wedge (S \text{ bounded below}) \Rightarrow \exists \inf S$



Proof

Denote $-S = \{-s : s \in S\}$.

①: S bounded below $\Rightarrow \exists\, m \in \mathbb{R}$ s.t. $m \le s \quad \forall s \in S \Rightarrow \forall s \in S\ -s \le -m$

$\Rightarrow -S$ is bounded above

②: $-S$ is bounded above $\overset{CA}{\Rightarrow} \exists \sup(-S) =: s_0$

③: $s_0 = \sup(-S) \iff (\forall s \in S\ \ -s \le s_0) \wedge (\forall M_1 < s_0\ \ \exists s_1 \in S\ (M_1 < -s_1))$

$\iff (\forall s \in S\ \ -s_0 \le s) \wedge (\forall \underset{\overset{\shortparallel}{m}}{-M_1} > -s_0\ \ \exists s_1 \in S\ (\underset{\overset{\shortparallel}{m}}{-M_1} > s_1))$

$\iff -s_0 = \inf S$

Archimedean Property

- $\forall\ a>0\ \exists\ n\in\mathbb{N}$ s.t. $\frac{1}{n}<a$

- $\forall\ b>0\ \exists\ n\in\mathbb{N}$ s.t $n>b$

Thm 4.6 (Archimedean Property)

$\forall\ a>0, b>0\ \exists\ n\in\mathbb{N}$ s.t. $na>b$

Proof: (by contradiction) Suppose AP is not true.

$\exists\ a>0, b>0$ s.t. $\forall\ n\in\mathbb{N}\quad na\leq b$

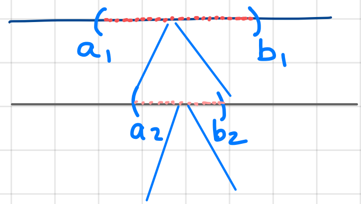
① $S:=\{an : n\in\mathbb{N}\}$ is bounded $\overset{CA}{\Longrightarrow}\ \exists\ \sup S =: s_0$

② Consider s_0-a: $\quad a>0 \Rightarrow s_0-a<s_0 \overset{\text{def.sup}}{\Longrightarrow} \exists\ n_0$ s.t. $s_0-a<an_0$

$\Rightarrow s_0<(n_0+1)a\in S \Rightarrow s_0\neq\sup S$, contradiction

$\blacksquare$

Denseness of $\mathbb{Q}$



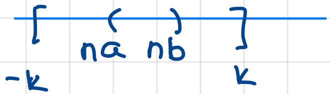
Thm 4.7 (Denseness of $\mathbb{Q}$)

$a<q<b$

$$(a,b\in\mathbb{R})\wedge(a<b)\Rightarrow \exists\, q\in\mathbb{Q}\ (q\in(a,b))$$

Proof: Enough to show that $\exists\ m\in\mathbb{Z},\, n\in\mathbb{N}$ s.t.

$$a<\frac{m}{n}<b \iff an<m<bn$$



① $b-a>0 \overset{AP}{\Longrightarrow} \exists\ n_0\in\mathbb{N}$ st. $n_0(b-a)>1$

How to show that $\exists\ m\in\mathbb{Z}$ s.t. $an_0<m<bn_0$?

Choose the smallest integer greater than an_0.

② $n_0\max\{|a|,|b|\}>0 \overset{AP}{\Longrightarrow} \exists\ k$ s.t. $k\geq n_0\max\{|a|,|b|\}$

$$\Rightarrow -k\leq n_0a\leq n_0b\leq k$$

③ $K:=\{j\in\mathbb{N}: -k\leq j\leq k,\ j>an_0\}$, K finite and $K\neq\emptyset\Rightarrow\exists\min K=:m$

④ $m=\min K\Rightarrow m-1\leq an_0\Rightarrow m\leq an_0+1\overset{①}{<}n_0b\Rightarrow n_0a<m<n_0b.$ $\blacksquare$