MATH 142A: Introduction to Analysis

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Today: Set of real numbers and completeness axiom

> Q&A: January 10

Next: Ross § 7

Week 2:

- homework 1 (due Friday, January 14)

Maximum and minimum

Let $\mathbb{F}$ be an ordered field and let $S \subset \mathbb{F}$, $S \neq \emptyset$

Def

Examples 1. Any finite nonempty subset of $\mathbb{F}$ has max and min

2. For $\mathbb{F} = \mathbb{R}$ and $a < b$, denote

$[a, b] :=$ $(a, b) :=$

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(a) $\max [a, b] = \max (a, b] =$ $\min [a, b] = \min [a, b) =$

Maximum and minimum

(b) $\max[a,b)$, $\max(a,b)$, $\min(a,b]$, $\min(a,b)$ do not exist

3. Recall $\max[0,\sqrt{2}] = \max\{x \in \mathbb{R} : 0 \le x \le \sqrt{2}\} =$

But $\max\{q \in \mathbb{Q} : 0 \le q \le \sqrt{2}\}$

Upper / lower bound

Let $\mathbb{F}$ be an ordered field and let $S \subset \mathbb{F}$, $S \neq \emptyset$

<u>Def</u> If , then M is called an

of S and S is called bounded above

If , then m is called a

of S and S is called bounded below

S is called , if it is bounded above and bounded below

<u>Examples</u> 1. Intervals $[a,b]$, $[a,b)$, $(a,b]$, (a,b) are bounded:

any $m \leq a$ is a lower bound, any $M \geq b$ is an upper bound

for these sets.

2. If $s_0 = \max S$, then any $M \geq s_0$ is an upper bound for S.

3. Sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ are not bounded above.

Supremum and infimum

Let $\mathbb{F}$ be an ordered field and let $S \subset \mathbb{F}$, $S \neq \emptyset$

Def If S is bounded above and S has a

then we call it the of S,

If S is bounded below and S has a

then we call it the of S,

Examples 1. If $\max S$ exists, then (similarly inf)

2. $\sup\,[a,b] = \sup[a,b) = \sup\,(a,b] = \sup\,(a,b) = b$ (similarly for inf)

Completeness axiom

3. (a) $\mathbb{F} = \mathbb{R}$ $\quad \max[0,\sqrt{2}] = \max\{x \in \mathbb{R} : 0 \le x \le \sqrt{2}\} =$

$\sup[0,\sqrt{2}] = \sup\{x \in \mathbb{R} : 0 \le x \le \sqrt{2}\} =$

(b) $\mathbb{F} = \mathbb{R}$ $\quad \max\{x \in \mathbb{Q} : 0 \le x \le \sqrt{2}\}$

$\sup\{x \in \mathbb{Q} : 0 \le x \le \sqrt{2}\} =$

(c) $\mathbb{F} = \mathbb{Q}$ $\quad \max\{x \in \mathbb{Q} : 0 \le x \le \sqrt{2}\}$

$\sup\{x \in \mathbb{Q} : 0 \le x \le \sqrt{2}\}$

Completeness Axiom

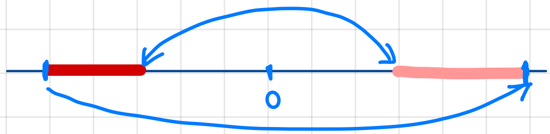
Every nonemply subset S of $\mathbb{R}$ that is bounded above has a least upper bound, i.e., $\sup S$ exists and is a real number.

Satisfied by $\mathbb{R}$ (by definition), not satisfied by $\mathbb{Q}$.

Corollary 4.5

Let $S \subset \mathbb{R}$.

Proof



Denote $-S = \{-s : s \in S\}$.

①: S bounded below $\Rightarrow$

②:

③:

Archimedean Property

- $\forall\ a>0\ \exists\, n\in\mathbb{N}$ s.t. $\frac{1}{n}<a$

- $\forall\ b>0\ \exists\, n\in\mathbb{N}$ s.t $n>b$

Thm 4.6 (Archimedean Property)

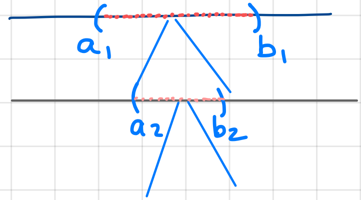
$\forall\ a>0,\ b>0\ \exists\, n\in\mathbb{N}$ s.t.

Proof: (by contradiction) Suppose AP is not true.

① $S:=\{an : n\in\mathbb{N}\}$

②

Denseness of $\mathbb{Q}$

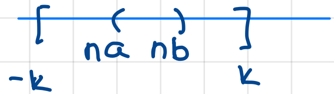


Thm 4.7 (Denseness of $\mathbb{Q}$)

$a<q<b$

$$(a,b\in\mathbb{R})\wedge(a<b) \Rightarrow \exists\, q\in\mathbb{Q}\ (q\in(a,b))$$

Proof: Enough to show that $\exists\, m\in\mathbb{Z},\ n\in\mathbb{N}$ s.t.



$$a<\frac{m}{n}<b \iff an<m<bn$$

①

How to show that $\exists\, m\in\mathbb{Z}$ s.t. $an_0<m<bn_0$?

Choose the smallest integer greater than an_0.

② $n_0\max\{|a|,|b|\}>0 \overset{AP}{\Rightarrow} \exists\, k$ s.t. $k\geq n_0\max\{|a|,|b|\}$

$$\Rightarrow -k\leq n_0a\leq n_0b\leq k$$

③ $K:=\{j\in\mathbb{N}: -k\leq j\leq k,\ j>an_0\}$, K finite and $K\neq\emptyset \Rightarrow \exists \min K=:m$

④ $m=\min K \Rightarrow m-1\leq an_0 \Rightarrow m\leq an_0+1 \overset{①}{<} n_0b \Rightarrow n_0a<m<n_0b.$ ■