MATH 142A: Introduction to Analysis

www.math.ucsd.edu/~ynemish/teaching/142a

Today: Sequences and their limits

> Q&A: January 12

Next: Ross § 9

Week 2:

- homework 1 (due Friday, January 14)

Symbols $+\infty$ and $-\infty$

Often it is convenient to work with $\mathbb{R} \cup \{+\infty, -\infty\}$

Extend $\leq$ to this set using rules:

- $\forall \; x \in \mathbb{R} \quad -\infty \leq x \leq +\infty$
- $-\infty \leq +\infty, \quad -\infty \leq -\infty, \quad +\infty \leq +\infty$

Use $\pm\infty$ to denote unbounded intervals

$$[a, +\infty) := \{x \in \mathbb{R} : x \geq a\}, \quad (a, +\infty) := \{x \in \mathbb{R} : x > a\}$$

$$(-\infty, b] := \{x \in \mathbb{R} : x \leq b\}, \quad (-\infty, b) := \{x \in \mathbb{R} : x < b\}$$

$$(-\infty, +\infty) := \mathbb{R}$$

We define
- $\sup S = +\infty$ if S is not bounded above
- $\inf S = -\infty$ if S is not bounded below

$$\sup \mathbb{N} = +\infty, \quad \inf \mathbb{Z} = -\infty$$

Sequences

Function, mapping: Let X and Y be two sets. We say that there is a function defined on X with values in Y, if via some rule f we associate to each element $x \in X$ an (one) element $y \in Y$. We write $f: X \to Y$, $x \overset{f}{\mapsto} y$ $(y = f(x))$

X is called the domain of definition of the function, $y = f(x)$ is called the image of x. E.g.: $f: [0,1) \to [0,1)$, $x \mapsto x^2$

Def (Sequence) A function $f: \mathbb{N} \to X \subset \mathbb{R}$, whose domain of definition is the set of natural numbers, is called a sequence.

$f(1), f(2), f(3), f(4), \dots$

$$\begin{array}{ccc} f(m), & f(m+1), & f(m+2), \dots \\ \| & \| & \| \\ f(m-1+1), & f(m-1+2), & f(m-1+3) \dots \end{array}$$

$g(n) := f(m-1+n)$

Notation: $(s_1, s_2, s_3, s_4, \dots) = (s_n)_{n=1}^{\infty}$

$(s_n)_{n \geq 1}$, $(s_n)_{n \in \mathbb{N}}$, $\{s_1, s_2, s_3, \dots\}$, $\{s_n\}_{n=1}^{\infty}$

$(s_m, s_{m+1}, s_{m+2}, \dots) = (s_n)_{n=m}^{\infty} = (s_n)_{n \geq m}$

Examples of sequences

- $(a_n)_{n=1}^{\infty}$, $a_n = 0$ $\qquad (a_n)_{n=1}^{\infty} = (0, 0, 0, 0, \dots)$
- $(a_n)_{n=1}^{\infty}$, $a_n = n$ $\qquad (a_n)_{n=1}^{\infty} = (1, 2, 3, 4, \dots)$
- $(a_n)_{n=1}^{\infty}$, $a_n = \frac{1}{n}$ $\qquad (a_n)_{n=1}^{\infty} = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$
- $b_n = \frac{1}{2^n}$, $n \in \{0, 1, 2, 3, \dots\}$ $\qquad (b_n)_{n=0}^{\infty} = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots)$
- $\sin\left(\frac{n\pi}{2}\right)$, $n \in \mathbb{N}$ $\qquad (1, 0, -1, 0, 1, 0, \dots)$
- $b_n = \left(1 + \frac{1}{n}\right)^n$, $n \in \mathbb{N}$ $\qquad (b_n)_{n=1}^{\infty} = (2, 2.25, 2.3704, 2.4414, 2.5216, \dots)$
- $a_n = n^2 \cdot \sin\left(\frac{1}{n^2}\right)$ $\qquad (a_n)_{n=1}^{\infty} = (0.84, 0.98, 0.997, 0.9993, 0.9997, \dots)$

Convergence

Def 7.1. A sequence (s_n) of real numbers is said to converge to the real number s if

$$\forall \varepsilon>0 \ \exists N\in\mathbb{N} \ \forall n>N \ \left(|s_n-s|<\varepsilon\right)$$

Notation: $\lim\limits_{n\to\infty} s_n = s$ or $s_n \to s$, $n\to\infty$

Def A sequence that does not converge is said to diverge.

convergent / divergent

Remark $N\in\mathbb{N}$ in the definition depends on ε.

$s_n = \frac{n+1}{n}$, $n\in\mathbb{N}$, $\lim\limits_{n\to\infty} s_n = 1$, $|s_n-n| = \left|\frac{n+1}{n}-1\right| = \frac{1}{n}$

$|s_n-s|<1$ for all $n>1$

$|s_n-s|<0.1$ for all $n>10$

$|s_n-s|<0.01$ for all $n>100$ (also for all $n>101$, $n>1000$)

Examples of sequences

- $(a_n)_{n=1}^{\infty}$, $a_n = 0$ $\quad (a_n)_{n=1}^{\infty} = (0,0,0,0,\dots)$ $\quad \lim\limits_{n\to\infty} a_n = 0$

- $(a_n)_{n=1}^{\infty}$, $a_n = n$ $\quad (a_n)_{n=1}^{\infty} = (1,2,3,4,\dots)$ $\quad$ diverges

- $(a_n)_{n=1}^{\infty}$, $a_n = \frac{1}{n}$ $\quad (a_n)_{n=1}^{\infty} = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$ $\quad \lim\limits_{n\to\infty} a_n = 0$

- $b_n = \frac{1}{2^n}$, $n \in \{0,1,2,\dots\}$ $\quad (b_n)_{n=0}^{\infty} = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots)$ $\quad \lim\limits_{n\to\infty} a_n = 0$

- $\sin\left(\frac{n\pi}{2}\right)$, $n \in \mathbb{N}$ $\quad (1, 0, -1, 0, 1, \dots)$ $\quad$ diverges

- $b_n = \left(1 + \frac{1}{n}\right)^n$, $n \in \mathbb{N}$ $\quad (b_n)_{n=1}^{\infty} = (2, 2.25, 2.3704, 2.4414, 2.5216, \dots)$

  $\lim\limits_{n\to\infty} b_n = e$

- $a_n = n^2 \sin\left(\frac{1}{n^2}\right)$, $n \in \mathbb{N}$ $\quad (a_n)_{n=1}^{\infty} = (0.84, 0.98, 0.997, 0.9993, 0.9997\dots)$

  $\lim\limits_{n\to\infty} n^2 \sin\left(\frac{1}{n^2}\right) = 1$

Uniqueness of limit

Prop. Let $(s_n)_{n=1}^{\infty}$ be a convergent sequence. Then

$$\lim_{n\to\infty} s_n = s \;\wedge\; \lim_{n\to\infty} s_n = t \;\Rightarrow\; s = t$$

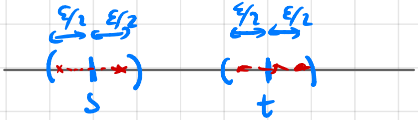
Proof. Fix $\varepsilon > 0$. Then

① $\lim_{n\to\infty} s_n = s \Rightarrow \exists N_1 \; \forall n > N_1 \quad |s_n - s| < \frac{\varepsilon}{2}$

② $\lim_{n\to\infty} s_n = t \Rightarrow \exists N_2 \; \forall n > N_2 \quad |s_n - t| < \frac{\varepsilon}{2}$

③ ① and ② $\Rightarrow \forall \; n > \max\{N_1, N_2\}$

$$|s-t| \overset{\text{Tr. Ineq.}}{\leq} |s_n - s| + |s_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$



$\Rightarrow \quad |s-t| < \varepsilon$ for any $\varepsilon > 0 \Rightarrow |s-t| = 0 \Rightarrow s = t$

$\blacksquare$

Example

Let $p \in \mathbb{Z}$. Then $\lim\limits_{n\to\infty} n^p = \begin{cases} 0, & p<0 & \text{(a)} \\ 1, & p=0 & \text{(b)} \\ \text{diverges}, & p>0 & \text{(c)} \end{cases}$

Proof (b) $n^0 = 1 \Rightarrow \forall \varepsilon > 0 \ \forall n \in \mathbb{N} \quad |n^0 - 1| = 0 < \varepsilon$. (take $N=1$)

(c) Suppose $\exists\, s \in \mathbb{R}$ s.t. $\lim\limits_{n\to\infty} n^p = s$. Then (take $\varepsilon = 1$)

$\exists N \in \mathbb{N} \ \forall n > N \quad |n^p - s| < 1$

$\Rightarrow \forall n > N \quad n^p < 1+s \Rightarrow \forall n > N \quad n < \sqrt[p]{1+s}$

(contradiction AP)

$\Rightarrow n^p$ is divergent

(a) Fix $\varepsilon > 0$, denote $q = -p \in \mathbb{N}$. $\{$find N s.t. $\forall n > N \ \frac{1}{n^q} < \varepsilon \ (\Leftrightarrow n^q > \frac{1}{\varepsilon}$

Take $N = \left[\sqrt[q]{\frac{1}{\varepsilon}}\right]$. Then for $n > N$

$$n^q > \frac{1}{\varepsilon} \Leftrightarrow \frac{1}{n^q} < \varepsilon \Leftrightarrow n^p < \varepsilon \Leftrightarrow |n^p - 0| < \varepsilon$$

$\blacksquare$

Example

$$\lim_{n\to\infty} \frac{5n^4-n-10}{7n^4-n^2} = \frac{5}{7}$$

$$\frac{5n^4-n-10}{7n^4-n^2} = \frac{\cancel{n^4}\left(5-\frac{1}{n^3}-\frac{10}{n^4}\right)}{\cancel{n^4}\left(7-\frac{1}{n^2}\right)}$$

$$\left|\frac{5n^4-n-10}{7n^4-n^2}-\frac{5}{7}\right| = \left|\frac{\cancel{35n^4}-7n-70-\cancel{35n^4}-5n^2}{49n^4-7n^2}\right| = \left|\frac{-5n^2-7n-70}{49n^4-7n^2}\right| < \varepsilon$$

$$\forall n>10 \quad 5n^2+\overset{\overset{n^2}{\vee}}{7n}+\overset{\overset{n^2}{\vee}}{70} < 7n^2 \Rightarrow \left|\frac{-5n^2-7n-70}{7n^2(7n^2-1)}\right| < \frac{7\cancel{n^2}}{\cancel{7n^2}(7n^2-1)} = \frac{1}{7n^2-1} < \frac{1}{n^2} < \varepsilon$$

$$N = \max\left\{10, \left[\frac{1}{\sqrt{\varepsilon}}\right]\right\}$$

Proof Fix $\varepsilon>0$. Then $\forall n > \max\left\{10, \left[\frac{1}{\sqrt{\varepsilon}}\right]\right\} = N$

$$\left|\frac{5n^4-n-10}{7n^4-n^2}-\frac{5}{7}\right| = \left|\frac{-5n^2-7n-10}{7n^2(7n^2-1)}\right| < \frac{7n^2}{7n^2(7n^2-1)} < \frac{1}{6n^2} < \frac{1}{n^2},$$

and $n > \left[\frac{1}{\sqrt{\varepsilon}}\right] \Rightarrow n > \frac{1}{\sqrt{\varepsilon}} \Rightarrow n^2 > \frac{1}{\varepsilon} \Rightarrow \frac{1}{n^2} < \varepsilon$ ■