MATH 142A: Introduction to Analysis

www.math.ucsd.edu/~ynemish/teaching/142a

Today: Sequences and their limits

> Q&A: January 12

Next: Ross § 9

Week 2:

- homework 1 (due Friday, January 14)

Symbols $+\infty$ and $-\infty$

Often it is convenient to work with $\mathbb{R} \cup \{+\infty, -\infty\}$.

Extend $\leq$ to this set using rules:

- $\forall \ x \in \mathbb{R}$
-

Use $\pm\infty$ to denote unbounded intervals

$[a, +\infty) :=$, $(a, +\infty) :=$

$(-\infty, b] :=$, $(-\infty, b) :=$

$(-\infty, +\infty) :=$

We define
- $\sup S =$ if S is not bounded above
- $\inf S =$ if S is not bounded below

$\sup \mathbb{N} =$, $\inf \mathbb{Z} =$

Sequences

Function, mapping: Let X and Y be two sets. We say that there is a function defined on X with values in Y, if via some rule f we associate to each element $x \in X$ an (one) element $y \in Y$. We write

X is called the domain of definition of the function,

$y = f(x)$ is called the image of x.

Def (Sequence) A function , whose domain of definition is the set of natural numbers, is called a sequence.

Notation:

Examples of sequences

-
-
-
-
-
-
-

Convergence

Def 7.1. A sequence (s_n) of real numbers is said to converge

to the real number s if

Notation: or

Def A sequence that does not converge is said to diverge.

convergent / divergent

Remark $N \in \mathbb{N}$ in the definition depends on ε.

$s_n =$

$|s_n - s| < 1$ for all

$|s_n - s| < 0.1$ for all

$|s_n - s| < 0.01$ for all

Examples of sequences

- $(a_n)_{n=1}^{\infty}$, $a_n = 0$ $\qquad (a_n)_{n=1}^{\infty} = (0, 0, 0, 0, \dots)$
- $(a_n)_{n=1}^{\infty}$, $a_n = n$ $\qquad (a_n)_{n=1}^{\infty} = (1, 2, 3, 4, \dots)$
- $(a_n)_{n=1}^{\infty}$, $a_n = \frac{1}{n}$ $\qquad (a_n)_{n=1}^{\infty} = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$
- $b_n = \frac{1}{2^n}$, $n \in \{0, 1, 2, \dots\}$ $\qquad (b_n)_{n=0}^{\infty} = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots)$
- $\sin\left(\frac{n\pi}{2}\right)$, $n \in \mathbb{N}$ $\qquad (1, 0, -1, 0, 1, \dots)$
- $b_n = \left(1 + \frac{1}{n}\right)^n$, $n \in \mathbb{N}$ $\qquad (b_n)_{n=1}^{\infty} = (2, 2.25, 2.3704, 2.4414, 2.5216, \dots)$
- $a_n = n^2 \sin\left(\frac{1}{n^2}\right)$, $n \in \mathbb{N}$ $\qquad (a_n)_{n=1}^{\infty} = (0.84, 0.98, 0.997, 0.9993, 0.9997 \dots)$

Uniqueness of limit

Prop. Let $(s_n)_{n=1}^{\infty}$ be a convergent sequence. Then

$$\lim_{n\to\infty} s_n = s \quad \wedge \quad \lim_{n\to\infty} s_n = t \quad \Rightarrow$$

Proof. Fix $\varepsilon > 0$. Then

1. $\lim_{n\to\infty} s_n = s \Rightarrow$

2. $\lim_{n\to\infty} s_n = t \Rightarrow$

3. 1 and 2 $\Rightarrow \forall\ n > \max\{N_1, N_2\}$

$\Rightarrow$

$\blacksquare$

Example

Let $p \in \mathbb{Z}$. Then $\lim\limits_{n\to\infty} n^p = \begin{cases} & p<0 & \text{(a)} \\ & p=0 & \text{(b)} \\ & , \; p>0 & \text{(c)} \end{cases}$

Proof (b) $n^0 = 1 \Rightarrow \forall \varepsilon>0 \ \forall n\in\mathbb{N} \quad |n^0 - 1| = 0 < \varepsilon$.

(c) Suppose $\exists\, s\in\mathbb{R}$ s.t. $\lim\limits_{n\to\infty} n^p = s$. Then

$\forall \varepsilon>0 \ \exists\, N\in\mathbb{N} \ \forall n>N \quad |n^p - s| < \varepsilon$

$\Rightarrow$

$\Rightarrow$ n^p is divergent

(a) Fix $\varepsilon>0$, denote $q=-p\in\mathbb{N}$. $\{$ find N s.t. $\forall n>N$

Take . Then for $n>N$

Example

$$\lim_{n\to\infty} \frac{5n^4-n-10}{7n^4-n^2} =$$

$$\frac{5n^4-n-10}{7n^4-n^2} =$$

$$\left|\frac{5n^4-n-10}{7n^4-n^2} - \frac{5}{7}\right| = \left|\frac{35n^4-7n-70-35n^4-5n^2}{49n^4-7n^2}\right| = \left|\frac{-5n^2-7n-70}{49n^4-7n^2}\right| < \varepsilon$$

$\forall\, n>10$ $\Rightarrow \left|\dfrac{-5n^2-7n-70}{7n^2(7n^2-1)}\right| <$

$N =$

Proof Fix $\varepsilon>0$. Then $\forall\, n > \max\left\{10, \left[\frac{1}{\sqrt{\varepsilon}}\right]\right\}$

$$\left|\frac{5n^4-n-10}{7n^4-n^2} - \frac{5}{7}\right| = \left|\frac{-5n^2-7n-10}{7n^2(7n^2-1)}\right| < \frac{7n^2}{7n^2(7n^2-1)} < \frac{1}{6n^2} < \frac{1}{n^2},$$

and $n > \left[\frac{1}{\sqrt{\varepsilon}}\right] \Rightarrow$