MATH 142A: Introduction to Analysis

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Today: Limit theorems for sequences

> Q&A: January 14, 20

Next: Ross § 9

Week 2:

- homework 1 (due Friday, January 14)

Last time

Def 7.1. A sequence (s_n) of real numbers is said to converge to the real number s if

$$\forall \varepsilon>0 \ \exists N\in\mathbb{N} \ \forall n>N \ (|s_n - s|<\varepsilon)$$

$$\lim_{n\to\infty} s_n = s \quad , \quad s_n \to s \ , \ n\to\infty$$

Example

Let $p\in\mathbb{Z}$. Then
$$\lim_{n\to\infty} n^p = \begin{cases} 0, & p<0 & (a) \\ 1, & p=0 & (b) \\ \text{diverges}, & p>0 & (c) \end{cases}$$

$\frac{1}{n^q}$, $q>0$

Example

$$\lim_{n\to\infty} \frac{5n^4 - n - 10}{7n^4 - n^2} = \frac{5}{7}$$

Convergent sequences are bounded

Def (Bounded sequence).

A sequence (s_n) is said to be bounded if the set $\{s_n : n \in \mathbb{N}\}$ is bounded (i.e., $\exists\, M>0\ \forall n \in \mathbb{N}\ |s_n| < M$)

Thm 9.1 Let (s_n) be convergent. Then (s_n) is bounded

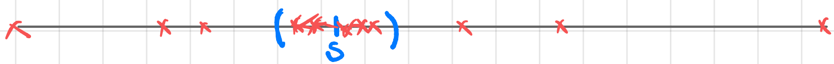
Proof. Let $s = \lim_{n\to\infty} s_n$, $s \in \mathbb{R}$. Then by Def. 7.1 $(\varepsilon = 1)$

$$\exists\, N\ \forall n > N \qquad |s_n - s| < 1$$

By the triangle inequality, $|s_n| \le |s| + |s_n - s|$

therefore $\forall n > N \quad |s_n| < |s| + 1$

Take $M = \max\{|s_1|, |s_2|, \dots, |s_N|, |s|+1\}$. Then $\forall n \in \mathbb{N}\ |s_n| \le M$ ∎



Multiplying covergent sequence by a scalar

Thm 9.2

Let (s_n) be convergent, $\lim_{n\to\infty} s_n = s \in \mathbb{R}$, and let $k \in \mathbb{R}$.

Then $\lim_{n\to\infty} k\cdot s_n = k\cdot s$ $\quad$ (i.e. $\lim_{n\to\infty} k\cdot s_n = k\cdot \lim_{n\to\infty} s_n$)

Proof. If $k=0$, then $\forall \varepsilon>0\ \forall n\in\mathbb{N}\ |k\cdot s_n| = 0 < \varepsilon$, and thus $\lim_{n\to\infty} k\cdot s_n = 0 = 0\cdot s$

Suppose $k\neq 0$. $\quad$ Fix $\varepsilon>0$ $\quad$ $\left\{ \begin{array}{l} \exists N\in\mathbb{N}\quad \forall n>N \\ |ks_n - ks| < \varepsilon \ (\Leftrightarrow |k|\,|s_n - s| < \varepsilon \Leftrightarrow |s_n - s| < \frac{\varepsilon}{|k|}) \end{array}\right.$

$\lim_{n\to\infty} s_n = s \Rightarrow \exists N\in\mathbb{N}\ \forall n>N\ \ |s_n - s| < \frac{\varepsilon}{|k|}$ (Def 7.1 with $\frac{\varepsilon}{|k|}$)

Then $\forall n>N \quad |ks_n - ks| = |k|\,|s_n - s| < |k|\cdot\frac{\varepsilon}{|k|} = \varepsilon$ $\quad\blacksquare$

Example

- $\lim_{n\to\infty} \frac{10}{n^4} = \lim_{n\to\infty} 10\cdot\frac{1}{n^4} = 10\cdot\lim_{n\to\infty}\frac{1}{n^4} = 10\cdot 0 = 0$
- $\forall k\in\mathbb{R}$, $\lim_{n\to\infty} k = \lim_{n\to\infty} k\cdot 1 = k\lim_{n\to\infty} 1 = k\cdot 1 = k$

Limit of a sum

Thm 9.3 Let (s_n) and (t_n) be two convergent sequences.

If $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$, then $\lim_{n\to\infty}(s_n+t_n) = s+t$ $\left(\lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n\right)$

Proof. Fix $\varepsilon>0$.

$\exists N\in\mathbb{N}\ \forall n>N\quad |s_n+t_n-(s+t)|<\varepsilon$

$|s_n-s+t_n-t| \overset{\text{Tr. Ineq.}}{\le} |s_n-s|+|t_n-t| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$

$\lim_{n\to\infty} s_n = s \Rightarrow \exists N_1\ \forall n>N_1\quad |s_n-s|<\frac{\varepsilon}{2}$

$\lim_{n\to\infty} t_n = t \Rightarrow \exists N_2\ \forall n>N_2\quad |t_n-t|<\frac{\varepsilon}{2}$

Then $\forall n>N:=\max\{N_1,N_2\}\quad |s_n+t_n-(s+t)| \le |s_n-s|+|t_n-t| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$

$\blacksquare$

Corollary $(s_n),(t_n)$ convergent $\Rightarrow \lim_{n\to\infty}(s_n-t_n) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} t_n$

Example $\lim_{n\to\infty}\left(5-\frac{1}{n^3}-\frac{10}{n^4}\right) = \lim_{n\to\infty} 5 - \lim_{n\to\infty}\frac{1}{n^3} - \lim_{n\to\infty}\frac{10}{n^4} = 5-0-0=5$

Limit of a product

Thm 9.4 Let (s_n) and (t_n) be convergent, $\lim\limits_{n\to\infty} s_n = s \in \mathbb{R}$, $\lim\limits_{n\to\infty} t_n = t \in \mathbb{R}$.

Then $\lim\limits_{n\to\infty} (s_n \cdot t_n) = s \cdot t = \left(\lim\limits_{n\to\infty} s_n\right)\left(\lim\limits_{n\to\infty} t_n\right)$

Proof Fix $\varepsilon > 0$.

$\exists N \ \forall n > N \ |s_n t_n - st| < \varepsilon$, can control $|s_n - s|$, $|t_n - t|$

$$|s_n t_n - st| = |s_n t_n - s_n t + s_n t - st| \le \underbrace{|s_n|\,|t_n - t|}_{\overset{?}{<}\ \varepsilon/2} + \underbrace{|t|\,|s_n - s|}_{\overset{?}{<}\ \varepsilon/2}$$

Thm 9.1

$\exists M > 0 : |s_n| \le M \Rightarrow |s_n|\,|t_n - t| \le M\,|t_n - t| < \frac{\varepsilon}{2}$

$|t| < |t| + 1 \Rightarrow |t|\,|s_n - s| < (|t|+1)\,|s_n - s| < \frac{\varepsilon}{2}$

$\exists N_1 \in \mathbb{N} \ \forall n > N_1 \ |s_n - s| < \dfrac{\varepsilon}{2(|t|+1)}$

$\exists N_2 \in \mathbb{N} \ \forall n > N_2 \ |t_n - t| < \dfrac{\varepsilon}{2M}$

$\Rightarrow$ $\forall n > \max\{N_1, N_2\} =: N$

$|s_n t_n - st| \le M\,|t_n - t| + |t|\,|s_n - s| < \varepsilon$

Example

$$\lim_{n\to\infty} \Big(\underset{\substack{\downarrow \\ 5}}{5 - \frac{1}{n^3} - \frac{10}{n^4}}\Big)\Big(\underset{\substack{\downarrow \\ 7}}{7 - \frac{1}{n^2}}\Big) = \lim_{n\to\infty}\Big(5 - \frac{1}{n^3} - \frac{10}{n^4}\Big) \lim_{n\to\infty}\Big(7 - \frac{1}{n^2}\Big) = 5 \cdot 7 = 35$$

$n \to \infty$

Limit of a sequence of reciprocals

Thm 9.5 Let (s_n) be a convergent sequence, $\lim_{n\to\infty} s_n = s$ such that $(\forall n \in \mathbb{N}\ (s_n \neq 0)) \wedge (s \neq 0)$. Then

$$\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s} = \frac{1}{\lim_{n\to\infty} s_n}$$

Proof. Fix $\varepsilon > 0$.

$\exists N \in \mathbb{N}\ \forall n > N \quad \left|\frac{1}{s_n} - \frac{1}{s}\right| < \varepsilon$

$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s_n - s}{s_n s}\right| = \frac{|s_n - s|}{|s_n||s|} \overset{?}{<} \varepsilon$

If $|s_n| \geq m > 0$, then $\left|\frac{1}{s_n} - \frac{1}{s}\right| \leq \frac{1}{|s|m} |s_n - s|$

$|s_n - s| < |s| m \varepsilon \Rightarrow \left|\frac{1}{s_n} - \frac{1}{s}\right| < \varepsilon$

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① $\exists m > 0 \ \inf\{|s_n| : n \in \mathbb{N}\} \geq m$. Proof①: $\exists N_1\ \forall n > N_1\ |s_n - s| < \frac{|s|}{2}$. Then

$\forall n > N_1 \quad |s_n| \overset{\text{Tr. Ineq.}}{\geq} |s| - |s_n - s| > |s| - \frac{|s|}{2} = \frac{|s|}{2}$. Take $m = \min\{|s_1|, |s_2|, \dots, |s_{N}|, \frac{|s|}{2}\} > 0$

Limit of a fraction of two convergent sequences

② $\exists N_2 \in \mathbb{N}$ $\forall n > N_2$ $|s_n - s| < |s| \cdot m \cdot \varepsilon$

$\forall n > N_2$ $$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s_n - s}{s_n s}\right| \leq \frac{|s_n - s|}{|s|\, m} < \frac{\varepsilon \cdot |s| \cdot m}{|s| \cdot m} = \varepsilon$$

$\blacksquare$

Thm 9.6.

Let (s_n), (t_n) be two convergent sequences, $\lim_{n\to\infty} s_n = s$, $\lim_{n\to\infty} t_n = t$, $\forall n \in \mathbb{N}$ $s_n \neq 0$, $s \neq 0$. Then

$$\lim_{n\to\infty} \frac{t_n}{s_n} = \frac{t}{s} = \frac{\lim\limits_{n\to\infty} t_n}{\lim\limits_{n\to\infty} s_n}.$$

Proof

$$\lim_{n\to\infty} \frac{t_n}{s_n} = \lim_{n\to\infty} t_n \cdot \frac{1}{s_n} \overset{\substack{\text{Thm 9.5} \\ \text{Thm 9.4}}}{=} \lim_{n\to\infty} t_n \cdot \lim_{n\to\infty} \frac{1}{s_n} \overset{\text{Thm 9.5}}{=} t \cdot \frac{1}{s} = \frac{t}{s}$$ $\blacksquare$

Examples

1) $$\lim_{n\to\infty} \frac{5n^4-n-10}{7n^4-n^2} = \frac{5}{7}$$

$$\lim_{n\to\infty} \frac{5n^4-n-10}{7n^4-n^2} \overset{\text{Thm 9.6}}{\neq} \frac{\lim_{n\to\infty}(5n^4-n-10) \to \text{diverges}}{\lim_{n\to\infty}(7n^4-n^2) \to \text{diverges}}$$

$$\parallel$$

$$\lim_{n\to\infty} \frac{\cancel{n^4}\left(5-\frac{1}{n^3}-\frac{10}{n^4}\right)}{\cancel{n^4}\left(7-\frac{1}{n^2}\right)} = \frac{\lim_{n\to\infty}\left(5-\frac{1}{n^3}-\frac{10}{n^4}\right)}{\lim_{n\to\infty}\left(7-\frac{1}{n^2}\right)} = \frac{5}{7}$$

2) $$\lim_{n\to\infty} \frac{5n^4-n-10}{7n^5-n^2} = \lim_{n\to\infty} \frac{\cancel{n^4}\left(5-\frac{1}{n^2}-\frac{10}{n^4}\right)}{n^5\left(7-\frac{1}{n^3}\right)} = \lim_{n\to\infty} \frac{5-\frac{1}{n^2}-\frac{10}{n^4}}{n\left(7-\frac{1}{n^3}\right)}$$

$$= \lim_{n\to\infty} \frac{1}{n} \cdot \frac{5-\frac{1}{n^2}-\frac{10}{n^4}}{7-\frac{1}{n^3}} = \lim_{n\to\infty} \frac{1}{n} \lim_{n\to\infty} \frac{5-\frac{1}{n^2}-\frac{10}{n^4}}{7-\frac{1}{n^3}} = 0 \cdot \frac{5}{7} = 0$$

Examples

3) $$\lim_{n \to \infty} \frac{5n^5 - n - 10}{7n^4 - n^2} =$$

$$= \frac{n^5\left(5 - \frac{1}{n^4} - \frac{10}{n^5}\right)}{n^4\left(7 - \frac{1}{n^2}\right)} = n \, \frac{5 - \frac{1}{n^4} - \frac{10}{n^4}}{7 - \frac{1}{n^2}}$$