

# MATH 142A: Introduction to Analysis

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Today: Limit theorems for sequences

> Q&A: January 19

Next: Ross § 10

Week 3:

- homework 2 (due Friday, January 21)

## Inequalities

### • Cauchy-Schwarz-Bunyakovsky inequality

Let  $n \in \mathbb{N}$  and  $\{a_1, \dots, a_n, b_1, \dots, b_n\} \subset \mathbb{R}$ . Then

$$\left( \sum_{k=1}^n a_k b_k \right)^2 \leq \sum_{k=1}^n a_k^2 \cdot \sum_{k=1}^n b_k^2 \quad (*)$$

Proof: Denote  $\sum_{k=1}^n a_k^2 =: A$ ,  $\sum_{k=1}^n b_k^2 =: B$ ,  $\sum_{k=1}^n a_k b_k =: C$ .

①  $A=0 \Rightarrow \forall k \ a_k=0 \Rightarrow C=0$   $0 \leq 0 \cdot B$ , (\*) holds

②  $A > 0$ . Consider the function  $p(x) := Ax^2 + 2Cx + B$

$$p(x) = \sum_{k=1}^n a_k^2 x^2 + 2a_k b_k x + b_k^2 = \sum_{k=1}^n (a_k x + b_k)^2 \geq 0 \Rightarrow 4C^2 - 4AB \leq 0 \\ \Rightarrow AB \geq C^2$$

Exercise  $\left( \sum_{k=1}^n a_k b_k \right)^2 = \sum_{k=1}^n a_k^2 \cdot \sum_{k=1}^n b_k^2$  ■

$$\Leftrightarrow \exists \lambda, \mu \in \mathbb{R}, |\lambda| + |\mu| \neq 0 \ \forall k \ \lambda a_k = \mu b_k.$$

# Inequalities

## AM-GM inequality:

Let  $n \in \mathbb{N}$ ,  $\{a_1, a_2, \dots, a_n\} \subset [0, +\infty)$ . Then

$$G_n := \sqrt[n]{a_1 a_2 \dots a_n} \leq \frac{a_1 + a_2 + \dots + a_n}{n} =: A_n$$

Proof ① If  $a_1 a_2 \dots a_n = 0$ , then  $G_n = 0$ ,  $A_n \geq 0$  ✓

② If  $n=1$ , then  $a_1 \leq a_1$  ✓

③ Suppose  $n > 1$  and  $\forall k \ a_k > 0$ . Then  $\forall k \ A_k > 0$  and

$\forall k \in \{2, 3, \dots, n\} \ \frac{A_k}{A_{k-1}} > 0$ , which is equivalent to  $\frac{A_k}{A_{k-1}} - 1 > -1 \ \forall k \in \{2, \dots, n\}$

Using Bernoulli's inequality:  $\left(\frac{A_k}{A_{k-1}}\right)^k \geq 1 + k \left(\frac{A_k}{A_{k-1}} - 1\right) = \frac{A_{k-1} + k A_k - k A_{k-1}}{A_{k-1}} = \frac{A_k}{A_{k-1}}$

$\Rightarrow A_k^k \geq A_{k-1}^{k-1} A_k$ . Apply for  $k = n, n-1, n-2, \dots, 2$

$$A_n^n \geq A_{n-1}^{n-1} A_n \geq A_{n-1}^{n-2} A_{n-1} A_n \geq \dots \geq A_2^2 A_1$$

# Inequalities

- Bernoulli's inequality (L1):

$$\forall a \geq -1 \quad \forall n \in \mathbb{N} \quad (1+x)^n \geq 1+nx$$

- Triangle inequality (L2):

$$\forall a, b \in \mathbb{R} \quad |a+b| \leq |a| + |b|$$

- Cauchy-Bunyakovsky-Schwarz inequality

Let  $n \in \mathbb{N}$  and  $\{a_1, \dots, a_n, b_1, \dots, b_n\} \subset \mathbb{R}$ . Then

$$\left( \sum_{k=1}^n a_k b_k \right)^2 \leq \sum_{k=1}^n a_k^2 \cdot \sum_{k=1}^n b_k^2$$

- AM-GM inequality

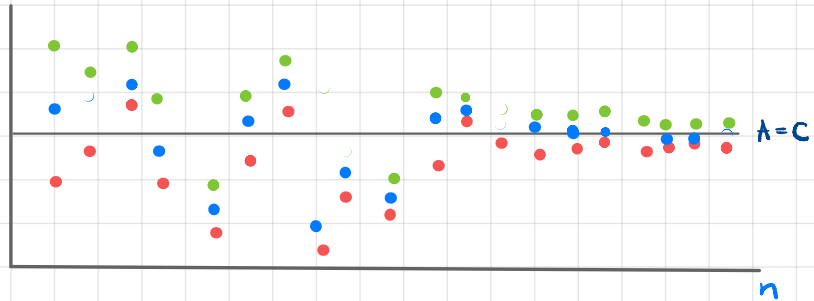
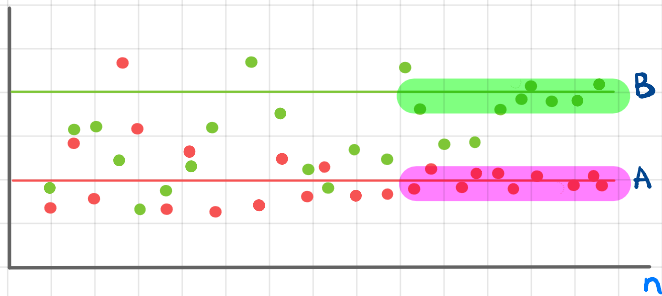
Let  $n \in \mathbb{N}$ ,  $\{a_1, a_2, \dots, a_n\} \subset [0, +\infty)$ . Then

$$G_n := \sqrt[n]{a_1 a_2 \dots a_n} \leq \frac{a_1 + a_2 + \dots + a_n}{n} =: A_n$$





# Limits and inequalities



Corollary 9.12 Suppose that  $\lim_{n \rightarrow \infty} a_n = A$ ,  $\lim_{n \rightarrow \infty} b_n = B$ .

(i)  $\exists N \forall n > N \ a_n > b_n \Rightarrow A \geq B$  (e.g.  $\forall n \ \frac{1}{n} > 0$ ,  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ )

(ii)  $\exists N \forall n > N \ a_n \geq b_n \Rightarrow A \geq B$

(iii)  $\exists N \forall n > N \ a_n > B \Rightarrow A \geq B$

(iv)  $\exists N \forall n > N \ a_n \geq B \Rightarrow A \geq B$

Proof: Exercise (for (i) and (ii) use proof by contradiction).

## Divergence to $\pm\infty$

Last time:  $\lim_{n \rightarrow \infty} \frac{5n^5 - n - 10}{7n^4 - n^2} = \lim_{n \rightarrow \infty} n \frac{5 - \frac{1}{n^4} - \frac{10}{n^5}}{7 - \frac{1}{n^2}} = ?$

Def 9.8. Let  $(s_n)$  be a sequence. We say that  $(s_n)$  diverges to  $+\infty$  ( $-\infty$ )

$$\lim_{n \rightarrow \infty} s_n = +\infty \text{ if } \forall M > 0 \exists N \in \mathbb{N} \forall n > N \quad s_n > M$$

$$\lim_{n \rightarrow \infty} s_n = -\infty \text{ if } \forall M < 0 \exists N \in \mathbb{N} \forall n > N \quad s_n < M$$

We say that  $(s_n)$  has a limit, if it converges, or diverges to  $+\infty$  or  $-\infty$ .

Example  $\lim_{n \rightarrow \infty} \frac{5n^5 - n - 10}{7n^4 - n^2} = +\infty$

Proof. Fix  $M > 0$ . If  $n > 1000$ , then  $\frac{5n^5 - n - 10}{7n^4 - n^2} > \frac{4n^5}{8n^4} = \frac{n}{2} [ > M ]$

Take  $N = \max\{1000, [2M]\}$ . Then  $\forall n > N$

$$\frac{5n^5 - n - 10}{7n^4 - n^2} > \frac{n}{2} > M$$

## Divergence to $\pm\infty$ and arithmetic operations

Thm 9.12 Let  $(s_n)$  be a sequence

$$(i) \lim_{n \rightarrow \infty} s_n = +\infty, k > 0 \Rightarrow \lim_{n \rightarrow \infty} (k \cdot s_n) = +\infty$$

$$(ii) \lim_{n \rightarrow \infty} s_n = +\infty \Leftrightarrow \lim_{n \rightarrow \infty} (-s_n) = -\infty$$

$$(iii) \lim_{n \rightarrow \infty} s_n = +\infty, k < 0 \Rightarrow \lim_{n \rightarrow \infty} (k \cdot s_n) = -\infty$$

Proof: Exercise

Thm 9.13 Let  $(s_n)$  and  $(t_n)$  be two sequences.

If  $\lim_{n \rightarrow \infty} s_n = +\infty$  and  $\inf\{t_n; n \in \mathbb{N}\} > -\infty$ , then  $\lim_{n \rightarrow \infty} (s_n + t_n) = +\infty$

Proof. Fix  $M > 0$  and denote  $m = \inf\{t_n; n \in \mathbb{N}\}$ .

$\lim_{n \rightarrow \infty} s_n = +\infty \Rightarrow \exists N \forall n > N \ s_n > M - m$ . Then  $\forall n > N \ \text{sur } t_n > s_n + m > M - m + m = M$

Examples

- $\lim_{n \rightarrow \infty} (n + \frac{1}{n}) = +\infty$
- $\lim_{n \rightarrow \infty} (n + \frac{1}{n})^2 - n^2 = 2$
- $\lim_{n \rightarrow \infty} (n^2 - n) = +\infty$
- $\lim_{n \rightarrow \infty} (n - n^2) = -\infty$

## Divergence to $\pm\infty$ and arithmetic operations

Thm 9.9 Let  $(s_n)$  and  $(t_n)$  be sequences such that

$$\lim_{n \rightarrow \infty} s_n = +\infty \text{ and } \left( \lim_{n \rightarrow \infty} t_n = t > 0 \text{ or } \lim_{n \rightarrow \infty} t_n = +\infty \right)$$

Then  $\lim_{n \rightarrow \infty} (s_n t_n) = +\infty$

Proof (For  $\lim_{n \rightarrow \infty} t_n = t > 0$ ) Fix  $M > 0$ . By Thm 9.11  $\exists N_1 \forall n > N_1, t_n > \frac{t}{2} > 0$

$$\lim_{n \rightarrow \infty} s_n = +\infty \Rightarrow \exists N_2 \forall n > N_2, s_n > M \cdot \frac{2}{t}. \text{ Then}$$

$$\forall n > \max\{N_1, N_2\}, s_n t_n > s_n \cdot \frac{t}{2} > M \cdot \frac{2}{t} \cdot \frac{t}{2} = M \quad \blacksquare$$

Thm 9.10 Let  $(s_n)$  be a sequence such that  $\forall n, s_n > 0$ . Then

$$\lim_{n \rightarrow \infty} s_n = +\infty \Leftrightarrow \lim_{n \rightarrow \infty} \left( \frac{1}{s_n} \right) = 0$$

Proof. ( $\Rightarrow$ ) Suppose  $\lim_{n \rightarrow \infty} s_n = +\infty$ . Fix  $\varepsilon > 0$ . [ $\exists N \forall n > N, \frac{1}{s_n} < \varepsilon \Leftrightarrow s_n > \frac{1}{\varepsilon}$ ]

$$\lim_{n \rightarrow \infty} s_n = +\infty \Rightarrow \exists N \forall n > N, s_n > \frac{1}{\varepsilon}. \text{ Then}$$

$$\forall n > N, \left| \frac{1}{s_n} - 0 \right| = \frac{1}{s_n} < \varepsilon \quad \blacksquare$$

( $\Leftarrow$ ) Exercise.

## Important examples

1. If  $q \in \mathbb{N}$ , then  $\lim_{n \rightarrow \infty} \frac{1}{n^q} = 0$  (L4)

2. If  $|a| < 1$ , then  $\lim_{n \rightarrow \infty} a^n = 0$

Proof. (1) If  $a = 0$ , then  $a^n = 0$ ,  $\lim_{n \rightarrow \infty} 0 = 0$

(2) Let  $a \neq 0$ . Fix  $\varepsilon > 0$ .  $[\exists N \forall n > N \quad |a^n - 0| = |a^n| = |a|^n < \varepsilon]$

Denote  $b = \frac{1}{|a|} - 1 > 0$  (so that  $|a| = \frac{1}{1+b}$ )

By Bernoulli's inequality

$$\left(\frac{1}{|a|}\right)^n = (1+b)^n \geq 1+nb > nb$$

and thus  $|a^n - 0| = |a^n| < \frac{1}{nb}$  [enough  $\frac{1}{nb} < \varepsilon \Leftrightarrow n > \frac{1}{\varepsilon b}$ ]

Take  $N = \lceil \frac{1}{\varepsilon b} \rceil$ . Then  $\forall n > N \quad |a^n - 0| < \frac{1}{nb} < \varepsilon$  ■

## Important examples

3.  $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$

Proof. ①  $\forall n \sqrt[n]{n} \geq 1$

② Write  $n = \sqrt[n]{n} \cdot \sqrt[n]{n} \cdot \overbrace{1 \cdot 1 \cdots 1}^{n-2}$ . Then by AM-GM inequality

$$\sqrt[n]{n} = \sqrt[n]{\underbrace{\sqrt[n]{n} \cdot \sqrt[n]{n} \cdot 1 \cdots 1}_{n-2}} \leq \frac{\sqrt[n]{n} + \sqrt[n]{n} + \overbrace{1 + \cdots + 1}^{n-2}}{n} = \frac{2\sqrt[n]{n} + n - 2}{n} = 1 + \frac{2}{n} - \frac{2}{n}$$

③  $\lim_{n \rightarrow \infty} \sqrt[n]{n} = +\infty$  [By AP  $\forall M > 0 \exists N \forall n > N \ n > M^2 \Leftrightarrow \sqrt[n]{n} > M$ ]

④ By Thm 9.10,  $\lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{n}} = 0 \Rightarrow \lim_{n \rightarrow \infty} \left(1 + \frac{2}{n} - \frac{2}{n}\right) = 1$

⑤  $1 \leq \sqrt[n]{n} \leq 1 + \frac{2}{n} - \frac{2}{n}$ , by Thm 9.11 and ④  $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$ .



## Important examples

$$4. \forall a > 0 \quad \lim_{n \rightarrow \infty} \sqrt[n]{a} = 1$$

Proof. If  $a = 1$ , then  $\lim_{n \rightarrow \infty} 1 = 1$

If  $a > 1$ , then ①  $\forall n \quad \sqrt[n]{a} \geq 1$

$$\text{② } \forall n > a \quad \sqrt[n]{a} \leq \sqrt[n]{n}$$

$$\text{③ By Thm 9.11 } \lim_{n \rightarrow \infty} \sqrt[n]{a} = 1$$

If  $a < 1$ , denote  $b = \frac{1}{a} > 1$ . Then

$$\text{① } \sqrt[n]{a} = \frac{1}{\sqrt[n]{b}} \quad \left| \begin{array}{l} \text{Thm 9.5} \\ \Rightarrow \end{array} \right.$$

$$\text{② } \lim_{n \rightarrow \infty} \sqrt[n]{b} = 1$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{a} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{b}} = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{b}} = \frac{1}{1} = 1.$$

