

# MATH 142A: Introduction to Analysis

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Today: Limit theorems for sequences

> Q&A: January 19

Next: Ross § 10

Week 3:

- homework 2 (due Friday, January 21)

## Inequalities

### • Cauchy-Schwarz-Bunyakovsky inequality

Let  $n \in \mathbb{N}$  and  $\{a_1, \dots, a_n, b_1, \dots, b_n\} \subset \mathbb{R}$ . Then

Proof: Denote  $\sum_{k=1}^n a_k^2 =: A$ ,  $\sum_{k=1}^n b_k^2 =: B$ ,  $\sum_{k=1}^n a_k b_k =: C$ .

①  $A = 0$

②  $A > 0$ . Consider the function  $p(x) :=$

Exercise  $\left( \sum_{k=1}^n a_k b_k \right)^2 = \sum_{k=1}^n a_k^2 \cdot \sum_{k=1}^n b_k^2$

# Inequalities

AM-GM inequality:

Let  $n \in \mathbb{N}$ ,  $\{a_1, a_2, \dots, a_n\} \subset [0, +\infty)$ . Then

Proof ① If  $a_1, a_2, \dots, a_n = 0$ , then

② If  $n=1$ , then

③ Suppose  $n > 1$  and  $\forall k \ a_k > 0$ . Then

# Inequalities

- Bernoulli's inequality (L1):

$$\forall a \geq -1 \quad \forall n \in \mathbb{N} \quad (1+x)^n \geq 1+nx$$

- Triangle inequality (L2):

$$\forall a, b \in \mathbb{R} \quad |a+b| \leq |a| + |b|$$

- Cauchy-Bunyakovsky-Schwarz inequality

Let  $n \in \mathbb{N}$  and  $\{a_1, \dots, a_n, b_1, \dots, b_n\} \subset \mathbb{R}$ . Then

$$\left( \sum_{k=1}^n a_k b_k \right)^2 \leq \sum_{k=1}^n a_k^2 \cdot \sum_{k=1}^n b_k^2$$

- AM-GM inequality

Let  $n \in \mathbb{N}$ ,  $\{a_1, a_2, \dots, a_n\} \subset [0, +\infty)$ . Then

$$G_n := \sqrt[n]{a_1 a_2 \dots a_n} \leq \frac{a_1 + a_2 + \dots + a_n}{n} =: A_n$$

## Limits and inequalities

Thm 9.11 (i) Let  $(a_n)$  and  $(b_n)$  be two convergent sequences,  $\lim_{n \rightarrow \infty} a_n = A$ ,  $\lim_{n \rightarrow \infty} b_n = B$ .

Then

(ii) Let  $(a_n), (b_n), (c_n)$  be three sequences such that  $\exists N_0 \forall n > N_0 \quad a_n \leq b_n \leq c_n$ .

Suppose that  $(a_n)$  and  $(c_n)$  are convergent,  $\lim_{n \rightarrow \infty} a_n = A$ ,  $\lim_{n \rightarrow \infty} c_n = C$

Then

Proof (i). Choose

Then

$$\lim_{n \rightarrow \infty} a_n = A \Rightarrow$$

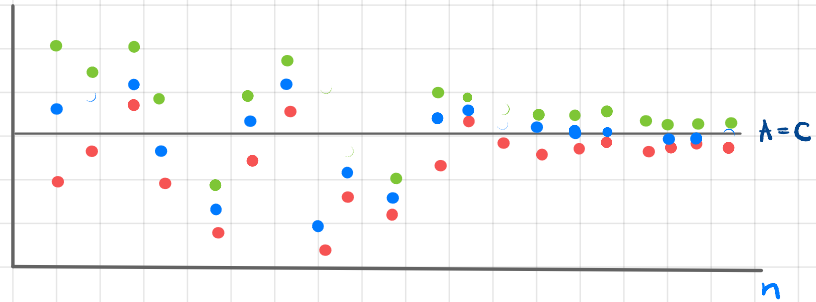
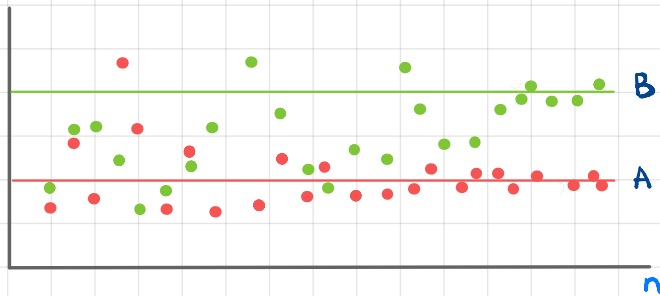
$$\lim_{n \rightarrow \infty} b_n = B \Rightarrow$$

(ii)

$$\lim_{n \rightarrow \infty} a_n = A \Rightarrow$$

$$\lim_{n \rightarrow \infty} c_n = C \Rightarrow .$$

# Limits and inequalities



Corollary 9.12 Suppose that  $\lim_{k \rightarrow \infty} a_n = A$ ,  $\lim_{k \rightarrow \infty} b_n = B$ .

(i)  $\exists N \forall n > N \ a_n > b_n$

(ii)  $\exists N \forall n > N \ a_n \geq b_n$

(iii)  $\exists N \forall n > N \ a_n > B$

(iv)  $\exists N \forall n > N \ a_n \geq B$

Proof: Exercise (for (i) and (ii) use proof by contradiction).

## Divergence to $\pm\infty$

Last time:  $\lim_{n \rightarrow \infty} \frac{5n^5 - n - 10}{7n^4 - n^2} = \lim_{n \rightarrow \infty} n \frac{5 - \frac{1}{n^4} - \frac{10}{n^5}}{7 - \frac{1}{n^2}} = ?$

Def 9.8. Let  $(s_n)$  be a sequence. We say that  $(s_n)$  diverges to  $+\infty$  ( $-\infty$ )

We say that  $(s_n)$  has a limit, if it converges, or diverges to  $+\infty$  or  $-\infty$ .

Example  $\lim_{n \rightarrow \infty} \frac{5n^5 - n - 10}{7n^4 - n^2} = +\infty$

Proof.

## Divergence to $\pm\infty$ and arithmetic operations

Thm 9.12 Let  $(s_n)$  be a sequence

(i)  $\lim_{n \rightarrow \infty} s_n = +\infty$ ,  $k > 0$

(ii)  $\lim_{n \rightarrow \infty} s_n = +\infty$

(iii)  $\lim_{n \rightarrow \infty} s_n = +\infty$ ,  $k < 0$

Proof: Exercise

Thm 9.13 Let  $(s_n)$  and  $(t_n)$  be two sequences.

If  $\lim_{n \rightarrow \infty} s_n = +\infty$  and  $\inf\{t_n; n \in \mathbb{N}\} > -\infty$ , then

Proof. Fix  $M > 0$  and denote  $m = \inf\{t_n; n \in \mathbb{N}\}$ .

$$\lim_{n \rightarrow \infty} s_n = +\infty \Rightarrow$$

Examples

- $\lim_{n \rightarrow \infty} (n + \frac{1}{n}) = +\infty$
- $\lim_{n \rightarrow \infty} (n + \frac{1}{n})^2 - n^2 = 2$
- $\lim_{n \rightarrow \infty} (n^2 - n) = +\infty$
- $\lim_{n \rightarrow \infty} (n - n^2) = -\infty$



## Divergence to $\pm\infty$ and arithmetic operations

Thm 9.9 Let  $(s_n)$  and  $(t_n)$  be sequences such that

$$\lim_{n \rightarrow \infty} s_n = +\infty \text{ and } \left( \lim_{n \rightarrow \infty} t_n = t > 0 \text{ or } \lim_{n \rightarrow \infty} t_n = +\infty \right)$$

Then

Proof (For  $\lim_{n \rightarrow \infty} t_n = t > 0$ ) Fix  $M > 0$ . By Thm 9.11

Thm 9.10 Let  $(s_n)$  be a sequence such that  $\forall n, s_n > 0$ . Then

Proof. ( $\Rightarrow$ ) Suppose  $\lim_{n \rightarrow \infty} s_n = +\infty$ .

( $\Leftarrow$ ) Exercise.

## Important examples

1. If  $q \in \mathbb{N}$ , then  $\lim_{n \rightarrow \infty} \frac{1}{n^q} = 0$  (L4)

2. If  $|a| < 1$ , then  $\lim_{n \rightarrow \infty} a^n =$

Proof. (1) If  $a = 0$ , then  $a^n = 0$ ,  $\lim_{n \rightarrow \infty} 0 = 0$

(2) Let  $a \neq 0$ . Fix  $\varepsilon > 0$ .

## Important examples

3.  $\lim_{n \rightarrow \infty} \sqrt[n]{n} =$

Proof. ①

② Write

③

④

⑤

## Important examples

$$4. \forall a > 0 \quad \lim_{n \rightarrow \infty} \sqrt[n]{a} =$$

Proof. If  $a = 1$ , then  $\lim_{n \rightarrow \infty} 1 = 1$

If  $a > 1$ , then ①

②

③

If  $a < 1$ , denote  $b = \frac{1}{a} > 1$ . Then

①

②