

MATH 142A: Introduction to Analysis

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Today: Monotone sequences

> Q&A: January 21

Next: Ross § 10

Week 3:

- Homework 2 (due Friday, January 21)
- Midterm 1 on Wednesday, January 26 (lectures 1-7)

Monotone sequences

Def 10.1 A sequence (S_n) is called

an **increasing sequence** if $\forall n \quad S_n \leq S_{n+1}$

a **decreasing sequence** if $\forall n \quad S_n \geq S_{n+1}$

a **monotone / monotonic sequence** if it is increasing or decreasing

Examples

$a_n = 0$ increasing and decreasing

$e_n = \frac{(-1)^n}{n}$ not monotonic

$b_n = n$ increasing

$f_n = \frac{n}{1+n}$ increasing
 $\frac{n+1}{n+2} - \frac{n}{n+1} = \frac{1}{(n+2)(n+1)} > 0$

$c_n = -n$ decreasing

$g_n = n^2 - 4n$ not monotone
-3, -4, -3, ...

$d_n = \frac{1}{n}$ decreasing

$h_n = \left(1 + \frac{1}{n}\right)^n$ increasing

Bounded monotone sequences converge

Thm 10.2 All bounded monotonic sequences converge.

Proof Let (s_n) be a bounded increasing sequence.

Denote $S := \{s_n : n \in \mathbb{N}\}$. Then

(s_n) bounded $\Rightarrow u := \sup S \in \mathbb{R}$

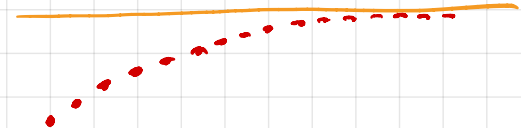
Fix $\varepsilon > 0$. Then

$$\textcircled{1} \quad u = \sup S \Rightarrow \forall n \quad s_n \leq u < u + \varepsilon \quad \exists n > m \leq n$$

$$\textcircled{2} \quad u = \sup S \Rightarrow \exists N \quad s_N > u - \varepsilon$$

(s_n) is increasing $\Rightarrow \forall n > N \quad u - \varepsilon < s_N \leq s_n$

$\textcircled{1} + \textcircled{2}$ imply that $\forall n > N \quad u - \varepsilon < s_n < u + \varepsilon$



Show $\lim_{n \rightarrow \infty} s_n = \sup S$

$\forall \varepsilon > 0 \exists N \forall n > N$

$$|s_n - u| < \varepsilon \Leftrightarrow u - \varepsilon < s_n < u + \varepsilon$$

Important example: the number e

Sequence $a_n = \left(1 + \frac{1}{n}\right)^n$ is convergent.

Proof. ① Sequence $b_n := \left(1 + \frac{1}{n}\right)^{n+1}$ is decreasing

$$\forall n \geq 2 \quad \frac{b_{n-1}}{b_n} = \frac{\left(1 + \frac{1}{n-1}\right)^n}{\left(1 + \frac{1}{n}\right)^{n+1}} = \frac{n}{n+1} \left(\frac{n}{n-1} \cdot \frac{n}{n+1}\right)^n = \frac{n}{n+1} \left(\frac{n^2}{n^2-1}\right)^n = \frac{n}{n+1} \left(1 + \frac{1}{n^2-1}\right)^n$$

Now use Bernoulli's inequality: $\left(1 + \frac{1}{n^2-1}\right)^n > 1 + n \cdot \frac{1}{n^2-1} > 1 + \frac{n}{n^2} > 1 + \frac{1}{n}$, so

$$\frac{b_{n-1}}{b_n} > \frac{n}{n+1} \cdot \frac{n+1}{n} = 1 \Rightarrow \forall n \geq 2 \quad b_{n-1} > b_n, (b_n) \text{ is decreasing.}$$

② Sequence b_n is bounded below: $\forall n \quad b_n > 0$

③ ① + ② + Thm. 10.2 \Rightarrow sequence $(b_n)_{n=1}^{\infty}$ converges

$$\textcircled{4} \quad \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{n+1} \cdot \frac{n}{n+1} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{n+1} =: e$$

Example

$$a_{n+1}^2 = 5 + a_n$$

Consider the sequence $(a_n)_{n=1}^{\infty}$ given by $a_1 = \sqrt{5}$, $a_{n+1} = \sqrt{5 + a_n}$
Is (a_n) convergent? If yes, what is the limit?

① (a_n) is monotonic ✓

① $\forall n \ a_n > 0$

$$a_{n+1} - a_n = \sqrt{5 + a_n} - \sqrt{5 + a_{n-1}} = \frac{5 + a_n - (5 + a_{n-1})}{\sqrt{5 + a_n} + \sqrt{5 + a_{n-1}}} = \frac{a_n - a_{n-1}}{\sqrt{5 + a_n} + \sqrt{5 + a_{n-1}}}$$

$$a_2 - a_1 = \sqrt{5 + \sqrt{5}} - \sqrt{5} > 0; \text{ if } a_n - a_{n-1} > 0, \text{ then } a_{n+1} - a_n > 0$$

Principle of math induction $\Rightarrow \forall n \ a_{n+1} - a_n > 0$, (a_n) is increasing

② (a_n) is bounded above ✓

$$a_{n+1} = \sqrt{5 + a_n} = \sqrt{5 \left(1 + \frac{a_n}{5}\right)} \stackrel{\text{AM-GM}}{\leq} \frac{5 + 1 + \frac{a_n}{5}}{2} = 3 + \frac{a_n}{10} \leq 3 + \frac{1}{10} \left(3 + \frac{a_{n-1}}{10}\right) \\ \leq 3 + \frac{3}{10} + \frac{3}{10^2} + \dots + \frac{3}{10^{n+1}} = 3 \left(1 + \frac{1}{10} + \frac{1}{10^2} + \dots + \frac{1}{10^{n+1}}\right) < 3 \cdot 2 = 6$$

① + ② + Thm. 10.2 $\Rightarrow \lim_{n \rightarrow \infty} a_n = A \in \mathbb{R}$. Define $b_n := a_{n+1} \cdot a_{n+1} - a_n - 5 = 0$

$$\lim_{n \rightarrow \infty} b_n = 0 \stackrel{\text{Thm 9.4, 9.3}}{=} A^2 - A - 5 \Rightarrow A \in \left\{ \frac{1 + \sqrt{21}}{2}, \frac{1 - \sqrt{21}}{2} \right\} \Rightarrow A = \frac{1 + \sqrt{21}}{2} \blacksquare$$

Unbounded monotone sequences

Thm 10.4 (i) If (s_n) is unbounded and increasing, then $\lim_{n \rightarrow \infty} s_n = +\infty$

(ii) If (s_n) is unbounded and decreasing, then $\lim_{n \rightarrow \infty} s_n = -\infty$

Proof (i) Fix $M > 0$. (s_n) unbounded $\Rightarrow \exists N \in \mathbb{N} \quad s_N > M$

(s_n) increasing $\Rightarrow \forall n > N \quad s_n \geq s_N > M$

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Corollary 10.5 If (s_n) is a monotone sequence, then it has a limit

i.e., (s_n) converges or diverges to $+\infty$ or diverges to $-\infty$.

limsup and liminf

Let (s_n) be convergent, $\lim_{n \rightarrow \infty} s_n = s$. Then $\forall \varepsilon > 0 \exists N$

$$\forall n > N |s_n - s| < \varepsilon \Leftrightarrow \forall n > N \quad s - \varepsilon < s_n < s + \varepsilon \Leftrightarrow \begin{aligned} \inf \{s_n : n > N\} &\geq s - \varepsilon \\ \sup \{s_n : n > N\} &\leq s + \varepsilon \end{aligned}$$

$$\lim_{n \rightarrow \infty} s_n = s \quad \text{iff} \quad \forall \varepsilon > 0 \exists N \quad s - \varepsilon \leq \inf \{s_n : n > N\} \leq \sup \{s_n : n > N\} \leq s + \varepsilon$$

\parallel u_N \parallel v_N

$$\left(\begin{array}{l} (u_n)_{n=1}^{\infty} \text{ is increasing} \\ (v_n)_{n=1}^{\infty} \text{ is decreasing} \end{array} \right) \Rightarrow \left(\begin{array}{l} \lim_{n \rightarrow \infty} s_n = s \Rightarrow \\ \lim_{n \rightarrow \infty} u_n = s \\ \lim_{n \rightarrow \infty} v_n = s \end{array} \right)$$

Def 10.6 Let (s_n) be a sequence. We define

$$\limsup_{n \rightarrow \infty} s_n = \overline{\lim}_{n \rightarrow \infty} s_n = \lim_{N \rightarrow \infty} \sup \{s_n : n > N\}$$

$$\liminf_{n \rightarrow \infty} s_n = \underline{\lim}_{n \rightarrow \infty} s_n := \lim_{N \rightarrow \infty} \inf \{s_n : n > N\}$$

If $\sup \{s_n : n \in \mathbb{N}\} = +\infty$, $\limsup s_n = +\infty$; if $\inf \{s_n : n \in \mathbb{N}\} = -\infty$, $\liminf s_n = -\infty$

limsup and liminf

Examples 1) $a_n = n, \forall N \quad \sup\{a_n : n > N\} = +\infty \Rightarrow \limsup_{n \rightarrow \infty} n = +\infty$

$\forall N \quad \inf\{a_n : n > N\} = N+1 \Rightarrow \liminf_{n \rightarrow \infty} n = +\infty$

2) $b_n = \frac{1}{n}, \forall N \quad \sup\{b_n : n > N\} = \frac{1}{N+1} \Rightarrow \limsup_{n \rightarrow \infty} \frac{1}{n} = 0$

$\forall N \quad \inf\{b_n : n > N\} = 0 \Rightarrow \liminf_{n \rightarrow \infty} \frac{1}{n} = 0$

3) $c_n = \frac{(-1)^n}{n} \forall N \quad \sup\{c_n : n > N\} \leq \frac{1}{N+1} \Rightarrow \limsup_{n \rightarrow \infty} \frac{(-1)^n}{n} \leq 0$

$\forall N \quad \inf\{c_n : n > N\} \geq -\frac{1}{N+1} \Rightarrow \liminf_{n \rightarrow \infty} \frac{(-1)^n}{n} \geq 0$

4) $d_n = (-1)^n \forall N \quad \sup\{d_n : n > N\} = 1 \Rightarrow \limsup_{n \rightarrow \infty} (-1)^n = 1$

$\forall N \quad \inf\{d_n : n > N\} = -1 \Rightarrow \liminf_{n \rightarrow \infty} (-1)^n = -1$

5) $e_n = n^{(-1)^n} \forall N \quad \sup\{e_n : n > N\} = +\infty \Rightarrow \limsup_{n \rightarrow \infty} n^{(-1)^n} = +\infty$

$\forall N \quad \inf\{e_n : n > N\} = 0 \Rightarrow \liminf_{n \rightarrow \infty} n^{(-1)^n} = 0$

Convergence and limsup/liminf

Thm. 10.7 Let (s_n) be a sequence in \mathbb{R} , $s \in \mathbb{R}$ or $s \in \{+\infty, -\infty\}$.

Then (i) $\lim_{n \rightarrow \infty} s_n = s \Rightarrow \liminf_{n \rightarrow \infty} s_n = \limsup_{n \rightarrow \infty} s_n = s$

(ii) $\limsup_{n \rightarrow \infty} s_n = \liminf_{n \rightarrow \infty} s_n = s \Rightarrow \lim_{n \rightarrow \infty} s_n = s$

Proof Denote $u_n = \inf \{s_n : n > N\}$, $v_n = \sup \{s_n : n > N\}$, $u = \liminf_{n \rightarrow \infty} s_n$, $v = \limsup_{n \rightarrow \infty} s_n$

(i) Three cases: $s = +\infty$, $s = -\infty$, $s \in \mathbb{R}$

$s = +\infty$ Fix $M > 0$. Then $\exists N \forall n > N \ s_n > 2M$, and this implies

that $\forall n > N \ u_n \geq u_N \geq 2M > M \Rightarrow \lim_{N \rightarrow \infty} u_N = \liminf_{n \rightarrow \infty} s_n = +\infty$

On the other hand, $\lim_{n \rightarrow \infty} s_n = +\infty \Rightarrow \forall n \ v_n = +\infty \Rightarrow \limsup_{n \rightarrow \infty} s_n = +\infty$

$s \in \mathbb{R}$ Fix $\varepsilon > 0$. Then $\exists N \forall n > N \ s - \frac{\varepsilon}{2} < s_n < s + \frac{\varepsilon}{2}$ Then

① $u_n \geq u_N \geq s - \frac{\varepsilon}{2} > s - \varepsilon \Rightarrow \forall n > N \ u_n > s - \varepsilon$ ② $v_n \leq v_N \leq s + \frac{\varepsilon}{2} < s + \varepsilon \Rightarrow \forall n > N \ v_n < s + \varepsilon$

③ $\forall n \ u_n \leq v_n$ ① + ② + ③ $\Rightarrow \forall n > N \ s - \varepsilon < u_n \leq v_n < s + \varepsilon$

Convergence and limsup/liminf

(ii) Three cases: $s = +\infty$, $s = -\infty$, $s \in \mathbb{R}$.

$$\boxed{s = +\infty}$$

$$\liminf_{n \rightarrow \infty} s_n = +\infty \Rightarrow \forall M > 0 \exists N \forall n > N \quad u_n > M$$

$$\text{Then } u_{N+1} = \inf \{s_n : n > N+1\} > M \Leftrightarrow \forall n > N+1 \quad s_n > M$$

$$\boxed{s \in \mathbb{R}}$$

$$\liminf_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} u_n = c, \quad \limsup_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} v_n = c$$

$$\forall n \geq 2 \quad u_{n-1} = \inf \{s_k : k > n-1\} \leq s_n \leq \sup \{s_k : k > n-1\} = v_{n-1}$$

$\downarrow \quad \downarrow \quad \downarrow$
 $n \rightarrow \infty \quad n \rightarrow \infty \quad n \rightarrow \infty$
 $s \quad s \quad s$

(Squeeze Lemma)

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