MATH 142A: Introduction to Analysis

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Today: Cauchy sequences

> Q&A: January 24

Next: Ross § 11

Week 4:

- Homework 3 (due Sunday, January 30)
- Midterm 1 on Wednesday, January 26 (lectures 1-7)
- Regrades of HW1 will be active on Gradescope on Tuesday, Jan 25

Cauchy sequences

Def 7.1. A sequence (s_n) of real numbers is said to converge to the real number s if

$$\forall \varepsilon>0 \ \exists N\in\mathbb{N} \ \forall n>N \ (|s_n-s|<\varepsilon)$$

Def. 10.8 A sequence (s_n) is called a Cauchy sequence if

$$\forall \varepsilon>0 \ \exists N\in\mathbb{N} \ \forall m,n>N \ (|s_m-s_n|<\varepsilon)$$

Examples Fix $\varepsilon>0$. $N>[\frac{1}{\varepsilon}]$

1. $a_n=\frac{1}{n}$: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ $\quad m,n>N \Rightarrow |a_m-a_n|=|\frac{1}{m}-\frac{1}{n}|<\frac{1}{N}<\varepsilon$

2. $b_n=\frac{(-1)^n}{n}$: $-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \cdots$ $\quad m,n>N \Rightarrow |b_m-b_n|\le|b_m|+|b_n|<\frac{1}{N}+\frac{1}{N}<\varepsilon$

3. $c_n=1+\frac{1}{n}$: $1, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \cdots$ $\quad m,n>N \Rightarrow |c_m-c_n|=|\frac{1}{m}-\frac{1}{n}|<\frac{1}{N}<\varepsilon$

Cauchy sequences

Lemma 10.9 Convergent sequences are Cauchy sequences.

Proof. Suppose $\lim_{n\to\infty} s_n = s \in \mathbb{R}$. Fix $\varepsilon > 0$.

Then $\exists N \; \forall n > N \; \left(|s_n - s| < \frac{\varepsilon}{2} \right)$

Using the triangle inequality, $\forall m, n \in \mathbb{N} \; \left(|s_n - s_m| \leq |s_n - s| + |s - s_m| \right)$

Therefore, $\forall m, n > N \; \left(|s_n - s_m| \leq |s_n - s| + |s_m - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \right)$

$\blacksquare$

Lemma 10.10 Cauchy sequences are bounded.

Proof. Suppose (s_n) is a Cauchy sequence. Then (take $\varepsilon = 1$)

$\exists N \in \mathbb{N} \; \forall m, n > N \; \left(|s_n - s_m| < 1 \right)$. Now take $m = N+1$

$\forall n > N \; |s_n - s_{N+1}| < 1 \Rightarrow \forall n > N \; |s_n| \leq |s_n - s_{N+1}| + |s_{N+1}| < |s_{N+1}| + 1$

With $M := \max\{ |s_1|, |s_2|, \ldots, |s_N|, |s_{N+1}| + 1 \}$ we have $\forall n \; |s_n| \leq M$.

$\blacksquare$

Cauchy sequences converge

Thm 10.11 (s_n) converges $\Leftrightarrow$ (s_n) is a Cauchy sequence

Proof. $(\Rightarrow)$ Lemma 10.9.

$(\Leftarrow)$ Suppose (s_n) is a Cauchy sequence.

By Lemma 10.10 (s_n) is bounded. Therefore, by Thm 10.7 it is enough to show that

$$\underbrace{\liminf_{n\to\infty} s_n}_{=\lim u_n} = \underbrace{\limsup_{n\to\infty} s_n}_{\lim v_n}$$

Denote $u_n = \inf\{s_k : k>n\}$, $v_n = \sup\{s_k : k>n\}$.

Fix $\varepsilon>0$. Then $\exists N\ \forall m,n>N\ \left(|s_m - s_n| < \frac{\varepsilon}{3}\right)$. In particular

$$\forall m,n>N\ \left(s_n < s_m + \frac{\varepsilon}{3}\right) \Rightarrow \forall m>N\ \ v_N \le s_m + \frac{\varepsilon}{3}$$

Similarly, $\forall m,n>N\ \left(s_n - \frac{\varepsilon}{3} < s_m\right) \Rightarrow \forall n>N\ \ u_N \ge s_n - \frac{\varepsilon}{3}$

Take $k>N$. Then $|v_k - u_k| = v_k - u_k \le v_N - u_N \le s_m + \frac{\varepsilon}{3} - s_n + \frac{\varepsilon}{3} < \varepsilon$

Therefore, $\lim_{k\to\infty}(v_k - u_k) = 0 = \lim_{k\to\infty} v_k - \lim_{k\to\infty} u_k = \limsup_{n\to\infty} s_n - \liminf_{n\to\infty} s_n$ ∎

Examples

1) Let $a_n = \frac{\cos(1)}{2} + \frac{\cos(2)}{2^2} + \frac{\cos(3)}{2^3} + \cdots + \frac{\cos(n)}{2^n}$. Then

(a_n) is a Cauchy sequence and thus (a_n) converges.

Proof. Fix $\varepsilon > 0$. Then $\forall\ m > n > N$

$$|a_m - a_n| = \left| \cancel{\frac{\cos(1)}{2} + \frac{\cos(2)}{2^2} + \cdots + \frac{\cos(n)}{2^n}} + \cdots + \frac{\cos(m)}{2^m} - \left(\cancel{\frac{\cos(1)}{2} + \cdots + \frac{\cos(n)}{2^n}} \right) \right|$$

$$= \left| \frac{\cos(n+1)}{2^{n+1}} + \cdots + \frac{\cos(m)}{2^m} \right| \leq \left| \frac{\cos(n+1)}{2^{n+1}} \right| + \cdots + \left| \frac{\cos(m)}{2^m} \right|$$

$$\leq \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^m} < \frac{1}{2^n} \quad [< \varepsilon]$$

$\exists N\ \forall n > N \left(\frac{1}{2^n} < \varepsilon \right) \Rightarrow \forall\ m, n > N\ |a_m - a_n| < \frac{1}{2^n} < \varepsilon \Rightarrow$ (a_n) is a Cauchy sequence. ∎

2) Let $b_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Take $\varepsilon = \frac{1}{2}$. Then $\forall n$

$$|b_{2n} - b_n| = \cancel{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}} + \frac{1}{n+1} + \cdots + \frac{1}{2n} - \cancel{\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)} = \overbrace{\frac{1}{n+1} + \cdots + \frac{1}{2n}}^{n}$$

$$> \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n} = \frac{n}{2n} = \frac{1}{2}$$

$\Rightarrow \forall N\ |b_{2(N+1)} - b_{N+1}| > \frac{1}{2} \Rightarrow (b_n)$ is not a Cauchy sequence $\Rightarrow$ does not converge. ∎

Asymptotic behavior of sequences

Lemma 10.12 (Exercise 9.12) (s_n) $(-1)^n$

Assume that all $s_n \neq 0$ and that $\lim_{n\to\infty} \left|\frac{s_{n+1}}{s_n}\right| = L \in [0, +\infty)$.

(a) If $L < 1$, then $\lim_{n\to\infty} s_n = 0$

(b) if $L > 1$, then $\lim_{n\to\infty} |s_n| = +\infty$ (Use part (a) and Thms 9.5 & 9.10)

Proof. Let $L \in [0,1)$. Fix $\varepsilon > 0$. Take $a \in (L, 1)$, $L < a < 1$.

Then by Thm 9.11 (i) (Lec 6) $\exists N \; \forall n > N \left(\left|\frac{s_{n+1}}{s_n}\right| < a \right)$

In particular, $|s_{N+2}| < a|s_{N+1}|$, $|s_{N+3}| < a|s_{N+2}| < a^2|s_{N+1}|$, ..., $|s_{N+k}| < a^{k-1}|s_{N+1}|$

Consider the sequence (b_n) with $b_n = a^n \frac{|s_{N+1}|}{a^{N+1}}$. Then

(i) by Thm 9.2 (Lec 5) and Important example 2 (Lec 6), $\lim_{n\to\infty} b_n = 0$

(ii) $\forall n \; 0 < |s_n| \leq b_n$, therefore by Thm 9.11 (ii) Lec 6 $\lim_{n\to\infty} |s_n| = 0$

Finally, $\forall n \; -|s_n| \leq s_n \leq |s_n| \Rightarrow \lim_{n\to\infty} s_n = 0$.

Example

Exercise 9.13.

$$\lim_{n\to\infty} a^n = \begin{cases} 0, & |a|<1 \\ 1, & a=1 \\ +\infty, & a>1 \\ \text{DNE}, & a\leq -1 \end{cases}$$

Proof. Case $|a|<1$: Consider the sequence $\left(\left|\frac{a^{n+1}}{a^n}\right|\right)_{n=1}^{\infty}$.

$$\forall n\in\mathbb{N} \quad \left|\frac{a^{n+1}}{a^n}\right| = |a| \Rightarrow \lim_{n\to\infty}\left|\frac{a^{n+1}}{a^n}\right| = |a| < 1 \overset{L10.12}{\Longrightarrow} \lim_{n\to\infty} a^n = 0$$

Case $a=1$: $\forall n \quad a^n = 1 \Rightarrow \lim_{n\to\infty} a^n = \lim_{n\to\infty} 1 = 1$

Case $a>1$: $\lim_{n\to\infty}\left|\frac{a^{n+1}}{a^n}\right| = \lim_{n\to\infty} a = a > 1 \overset{L10.12}{\Longrightarrow} \lim_{n\to\infty} |a^n| = \lim_{n\to\infty} a^n = +\infty.$

Case $a\leq -1$: Denote $b=-a\geq 1$, so that $a^n = (-1)^n \cdot b^n$, Note $\forall n\ b^n \geq 1$

Then $\forall N\in\mathbb{N}$
- $\exists n_1 > N$ (enough to take $n_1 = 2k > N$) $a^{n_1} = (-1)^{2k} b^{2k} \geq 1$
- $\exists n_2 > N$ (enough to take $n_2 = 2k+1 > N$) $a^{n_2} = (-1)^{2k+1} b^{2k+1} \leq -1$

Therefore, $\limsup_{n\to\infty} a^n = 1 \neq -1 = \liminf_{n\to\infty} a^n \Rightarrow (a^n)$ is divergent.

Important example 6 (asymptotic growth).

For any $p \in \mathbb{N}$ and any $a > 1$

$$\lim_{n\to\infty} \frac{n^p}{a^n} = 0$$

(exponential sequences grow to ∞ faster than polynomial sequences)

Proof. Denote $x_n := \frac{n^p}{a^n}$. Then

$$\frac{x_{n+1}}{x_n} = \frac{(n+1)^p}{a^{n+1}} \cdot \frac{a^n}{n^p} = \frac{1}{a}\left(\frac{n+1}{n}\right)^p = \frac{1}{a}\cdot\left(1+\frac{1}{n}\right)^p$$

① $\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1$

② By Thm 9.4 + ① (applied $p-1$ times) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^p = 1$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = \lim_{n\to\infty}\frac{1}{a}\cdot\left(1+\frac{1}{n}\right)^p = \frac{1}{a}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^p = \frac{1}{a} < 1$$

③ By Lemma 10.12, $\lim_{n\to\infty} x_n = \lim_{n\to\infty}\frac{n^p}{a^n} = 0$ ∎.

Important example 7 (asymptotic growth).

For any $a > 1$

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0$$

(factorial grows to ∞ faster than any exponential sequence)

Proof. Denote $y_n := \dfrac{a^n}{n!}$

① $$\lim_{n\to\infty} \left|\frac{y_{n+1}}{y_n}\right| = \lim_{n\to\infty} \frac{a^{n+1}}{(n+1)!} \cdot \frac{n!}{a^n} = \lim_{n\to\infty} \frac{a}{n+1} = 0$$

② By Lemma 10.12, $\displaystyle\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{a^n}{n!} = 0.$

$\blacksquare$