

MATH 142A: Introduction to Analysis

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Today: Subsequences

> Q&A: Jan 28

Next: Ross § 11-12

Week 4:

- Homework 3 (due Sunday, January 30)

Subsequences

$$a_n = (-1)^n, n \geq 1 : \text{ } \overset{\text{yellow}}{-1}, \overset{\text{green}}{1}, \overset{\text{yellow}}{-1}, \overset{\text{green}}{1}, \overset{\text{yellow}}{-1}, \overset{\text{green}}{1}, \overset{\text{yellow}}{-1}, \overset{\text{green}}{1}, \dots$$

$$n_k = 2k-1, (a_{n_k}) = (-1, -1, -1, -1, \dots); n_k = 2k, (a_{n_k}) = (1, 1, 1, 1, \dots)$$

$$b_n = \cos\left(\frac{\pi n}{2}\right), n \geq 1 : (0, \overset{\text{green}}{-1}, 0, \overset{\text{yellow}}{1}, 0, \overset{\text{pink}}{-1}, 0, \overset{\text{green}}{1}, 0, \overset{\text{pink}}{-1}, 0, \overset{\text{green}}{1}, \dots)$$

$$n_k = 2k-1, (b_{n_k}) = (0, 0, 0, 0); n_k = 3k, (b_{n_k}) = (\overset{\text{green}}{0}, \overset{\text{yellow}}{-1}, \overset{\text{pink}}{0}, 1, 0, \dots)$$

$$c_n = n, n \geq 1 : 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, \dots \quad \text{(~~2, 1, 4, 3, 6, 5, \dots~~)}$$

$$(n_k) = (1, 2, 3, 5, 7, 11, 13, \dots), (c_{n_k}) = (1, 2, 3, 5, 7, 11, 13, \dots)$$

$$d_n = \cos(n), n \geq 1 : \cos(1), \cos(2), \cos(3), \cos(4), \cos(5), \cos(6), \dots$$

$$(n_k) = (1, 7, 8, 9, 23, 24, 1002, \dots) \quad (d_{n_k}) = (\cos(1), \cos(7), \cos(8), \dots)$$

Def 11.1 Let (s_n) be a sequence of real numbers and let

$1 \leq n_1 < n_2 < \dots < n_k < \dots$ be an increasing sequence of natural numbers.

Then $(s_{n_k})_{k=1}^{\infty} = (s_{n_1}, s_{n_2}, s_{n_3}, \dots)$ is called a **subsequence** of $(s_n)_{n=1}^{\infty}$.

Subsequences

Thm 11.2 Let (s_n) be a sequence. Let $t \in \mathbb{R}$.

(i) There exists a (monotonic) subsequence of (s_n) converging to t
 $\Leftrightarrow \forall \varepsilon > 0$ the set $\{n \in \mathbb{N} : |s_n - t| < \varepsilon\}$ is infinite

Proof. (\Rightarrow) Exercise.

(\Leftarrow) $\forall \varepsilon > 0$ the set $\{n \in \mathbb{N} : |s_n - t| < \varepsilon\}$ is infinite.

Case 1: the set $\{n : s_n = t\}$ is infinite, take (s_{n_k}) with $s_{n_k} = t \quad \forall k$.

Case 2: $\forall \varepsilon > 0$ the set $\{n : 0 < |s_n - t| < \varepsilon\}$ is infinite.

Either (a) $\forall \varepsilon > 0$ $\{n : t - \varepsilon < s_n < t\}$ is infinite



or (b) $\forall \varepsilon > 0$ $\{n : t < s_n < t + \varepsilon\}$ is infinite

Consider Case 2(a). We want to construct an increasing subsequence that converges to t .

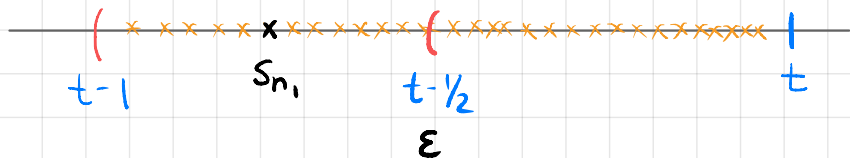
Proof of Thm 11.2 (i)

$$a_1 = 1, \quad a_{k+1} = 1 + a_k, \quad a_n = n$$

Suppose that $\forall \varepsilon > 0$ $\{n: t - \varepsilon < s_n < t\}$ is infinite

① Choose n_1 such that

$$t - 1 < s_{n_1} < t$$



② Take $\varepsilon = t - \max\{s_{n_1}, t - \frac{1}{2}\}$, so that $t - \varepsilon = \max\{s_{n_1}, t - \frac{1}{2}\}$

$\varepsilon > 0$ (since $s_{n_1} < t$, and $t - \frac{1}{2} < t$), and thus the set

$S_2 := \{n: \underbrace{\max\{s_{n_1}, t - \frac{1}{2}\}}_{t - \varepsilon} < s_n < t\}$ is infinite.

Choose $n_2 \in S$ such that $n_2 > n_1$. Then $\max\{s_{n_1}, t - \frac{1}{2}\} < s_{n_2} < t$.

Ⓚ Suppose we have numbers $n_1 < n_2 < \dots < n_{k-1}$ such that

$$\forall j \quad (\max\{s_{n_{j-1}}, t - \frac{1}{j}\} < s_{n_j} < t)$$

$$\text{Take } \varepsilon = t - \max\{s_{n_{k-1}}, t - \frac{1}{k}\}$$

$\{n: t - \varepsilon < s_n < t\}$ is infinite $\Rightarrow \exists n_k > n_{k-1}$ s.t. $\max\{s_{n_{k-1}}, t - \frac{1}{k}\} < s_{n_k} < t$

$(s_{n_k})_{k=1}^{\infty}$ is a subsequence of $(s_n)_{n=1}^{\infty}$, and $\forall k \quad t - \frac{1}{k} < s_{n_k} < t \Rightarrow \begin{matrix} s_{n_k} \rightarrow t \\ k \rightarrow \infty \end{matrix}$

Subsequences

Thm 11.2 Let (s_n) be a sequence.

(ii) (s_n) has a (monotonic) subsequence that diverges to $+\infty$

$\Leftrightarrow (s_n)$ is unbounded above

(iii) (s_n) has a (monotonic) subsequence that diverges to $-\infty$

$\Leftrightarrow (s_n)$ is unbounded below

Proof (ii) (\Rightarrow) Exercise.

(\Leftarrow) Suppose that (s_n) is unbounded above.

① Let $n_1 = 1$, so that $s_{n_1} = s_1$

② (s_n) unbounded above $\Rightarrow T_2 := \{n : \max\{s_1, 2\} < s_n\}$ is infinite

\vdots
choose $n_2 \in T_2$ s.t. $n_2 > n_1$

③ $T_k := \{n : \max\{s_{n_{k-1}}, k\} < s_n\}$ is infinite, choose $n_k \in T_k$ s.t. $n_k > n_{k-1}$

Then (s_{n_k}) is a subsequence, $\forall k \quad s_{n_k} > k \Rightarrow \lim_{k \rightarrow \infty} s_{n_k} = +\infty$ ■

Subsequences

Thm 11.3 If (s_n) converges, then any subsequence of (s_n) converges to the same limit.

Proof. Let (s_{n_k}) be a subsequence of (s_n) .

$$\textcircled{1} \quad \forall k \in \mathbb{N} \quad (n_k \geq k)$$

Proof by induction: $n_1 \geq 1$

if $n_{k-1} \geq k-1$, then $n_k \geq n_{k-1} + 1 \geq k$

(s_{n_k}) $\textcircled{2}$ Suppose (s_n) converges to $s \in \mathbb{R}$. Fix $\varepsilon > 0$. Then

$\exists N \in \mathbb{N} \quad \forall n > N \quad (|s_n - s| < \varepsilon)$. But since $\forall k \quad n_k \geq k$
 $\forall k > N \quad (n_k > N)$ and thus $(|s_{n_k} - s| < \varepsilon)$



Subsequences

Thm 11.4 Every sequence has a monotonic subsequence.

Proof. Let (s_n) be a sequence of real numbers.

We say that s_n is dominant if $\forall m > n \quad (s_n > s_m)$

Denote $D = \{n : s_n \text{ is dominant}\}$

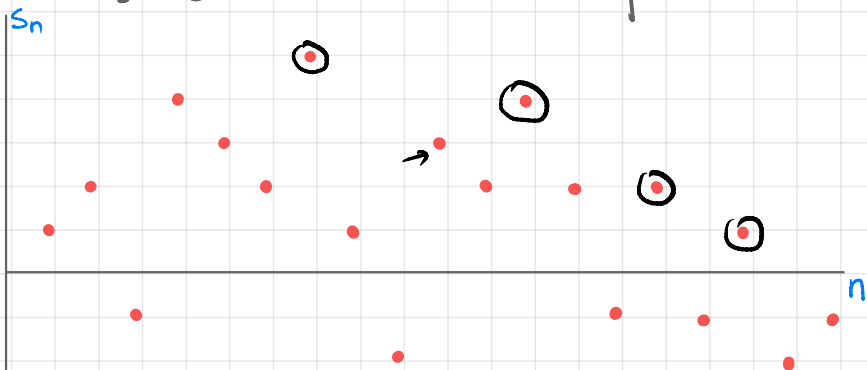
Case 1: D is infinite. Take $n_1 = \min D, \dots, n_k = \min \{n \in D : n > n_{k-1}\}$

Then $n_1 < n_2 \Rightarrow s_{n_1} > s_{n_2}, \quad n_{k-1} < n_k \Rightarrow s_{n_{k-1}} > s_{n_k} \Rightarrow (s_{n_k})$ is decreasing

Case 2: D is finite. Take $n_1 = \max D + 1$. Then s_{n_1} is not dominant

$\Rightarrow \exists n_2 > n_1$ s.t. $s_{n_2} \geq s_{n_1}$. Term s_{n_2} is not dominant $\Rightarrow \exists n_3 > n_2 (s_{n_3} \geq s_{n_2})$

If we have n_{k-1} , then $s_{n_{k-1}}$ is not dominant $\Rightarrow \exists n_k > n_{k-1} (s_{n_k} \geq s_{n_{k-1}})$
 $\Rightarrow (s_{n_k})$ is increasing



Bolzano-Weierstrass Theorem

Thm 11.5 Every bounded sequence has a convergent subsequence.

Proof Let (s_n) be a bounded sequence.

By Thm 11.4 (s_n) has a monotonic subsequence (s_{n_k})

Since (s_n) is bounded, (s_{n_k}) is also bounded.

(s_{n_k}) is monotonic and bounded, therefore by Thm 10.2

$(s_{n_k})_{k=1}^{\infty}$ converges.

