MATH 142A: Introduction to Analysis

www.math.ucsd.edu/~ynemish/teaching/142a

Today: Natural, rational, algebraic numbers > Q&A: January 5

Next: Ross § 3

Week 1:

- visit course website
- homework 0 (due Friday, January 7)
- join Piazza

Logical symbolism

Common logical connectives

$\neg$ negation ("not")

$\wedge$ and

$\vee$ or

$\Rightarrow$ implies

$\Leftrightarrow$ is equivalent to

Example:

A: Alice plays accordion

B: Bob reads a book

C: Alice and Bob stay at home

$C \wedge A \Rightarrow \neg B$

$(A \wedge \neg B) \vee (\neg A \wedge B)$

Typical mathematical statement: $A \Rightarrow B$

Typical proof: $(A \Rightarrow C_1) \wedge (C_1 \Rightarrow C_2) \wedge \cdots \wedge (C_n \Rightarrow B)$

$$A \Rightarrow C_1 \Rightarrow C_2 \Rightarrow \cdots \Rightarrow C_n \Rightarrow B$$

Logical symbolism

Basic rules for constructing proofs

- if A is true and $A \Rightarrow B$, then B is true
- the law of excluded middle: $A \vee \neg A$ is always true
  - ↳ used in proofs by contradiction
- rule of double negation: $\neg(\neg A) \Leftrightarrow A$

Use words instead of symbols (most of the time)

$A \Rightarrow B$	$A \Leftrightarrow B$
A implies B	A is equivalent to B
B follows from A	A if and only if B
B is necessary condition for A	A is necessary and sufficient for B
A is sufficient condition for B	

Logical symbolism

Think about the following statements

(a) $\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$

(b) $\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$

(c) $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$

Set theory notation

A set is a "collection of distinguishable objects"

- a set may consist of any distinguishable objects
- a set is uniquely determined by the collection of objects it consists of
- a set can be defined as a collection of objects having certain property

$\{x, y, z\}$ – listing objects

$\{x : P(x)\}$ – the set of all objects x that satisfy property P

$\{1, 2, 3, 4, 5\}$ or $\{n : n \in \mathbb{N} \text{ and } n \leq 5\}$

If S is a set, $x \in S$ means that x is an element of S

$x \notin S$ x is not an element of S

Set theory notation

S, T are two sets, then $T \subset S$ means that each element of T belongs to S.

$$\{1,2,3,4,5\} \subset \mathbb{N}, \quad \mathbb{N} \subset \mathbb{R}, \quad \mathbb{R} \subset \mathbb{R}$$

$$\{\{1,2\}, \{1,3\}\}$$

Defining a set from another set by specifying a rule

$$\left\{\frac{1}{n} : n \in \mathbb{N}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}$$

Operations on sets

If we have 2 sets S, T, then

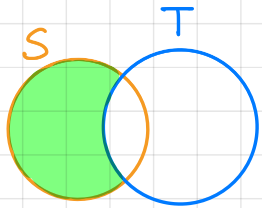
- $S \setminus T = \{x : (x \in S) \wedge (x \notin T)\}$ is the difference between S and T
- $S \cup T = \{x : (x \in S) \vee (x \in T)\}$ is the union of S and T
- $S \cap T = \{x : (x \in S) \wedge (x \in T)\}$ is the intersection of S and T

Set theory notation

$\mathcal{A}$ is a set, S_α, $\alpha \in \mathcal{A}$, is a collection of sets, then

$$\bigcup_{\alpha\in\mathcal{A}} S_\alpha = \{x : x \in S_\alpha \text{ for at least one } \alpha \in \mathcal{A}\}$$

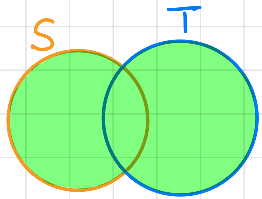
$$\bigcap_{\alpha\in\mathcal{A}} S_\alpha = \{x : x \in S_\alpha \text{ for all } \alpha \in \mathcal{A}\}$$



$S \setminus T$

Examples $S = \{1, 2, 4, 6\}$, $T = \{1, 3, 5\}$

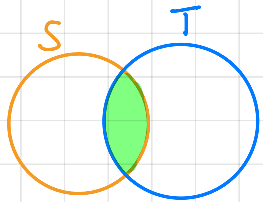
$S \setminus T = \{2, 4, 6\}$



$S \cup T$

$S \cup T = \{1, 2, 3, 4, 5, 6\}$

$S \cap T = \{1\}$



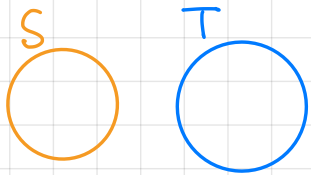
$S \cap T$

Empty set is the set with no elements, $\emptyset$

$S = \{1, 2, 3\}$

$T = \{4, 5, 6\}$

$S \cap T = \emptyset$



$S \cap T = \emptyset$

Natural numbers

We assume that we know what natural numbers are: numbers we use to count objects.

$$\{1, 2, 3, 4, \dots\} =: \mathbb{N}$$

Peano Axioms:

N1. $1 \in \mathbb{N}$

N2. $n \in \mathbb{N} \Rightarrow \overbrace{n+1}^{\text{successor}} \in \mathbb{N}$

N3. For any $n \in \mathbb{N}$, $n+1 = 1$ is false

N4. $(m, n \in \mathbb{N}) \wedge (m+1 = n+1) \Rightarrow m = n$

N5. $S \subset \mathbb{N} \wedge 1 \in S \wedge (n \in S \Rightarrow n+1 \in S) \Rightarrow S = \mathbb{N}$

Properties N1 – N5 define $\mathbb{N}$ uniquely.

Principle of mathematical induction

Let $P_1, P_2, P_3, \dots$ be a list of statements that may or may not be true. Then

$$\left.\begin{array}{ll} (I_1) & P_1 \text{ is true} \\ (I_2) & P_n \text{ is true} \Rightarrow P_{n+1} \text{ is true} \end{array}\right| \Rightarrow \begin{array}{l} \text{all statements } P_1, P_2, P_3, \dots \\ \text{are true} \end{array}$$

(I_1) basis of induction $\qquad$ (I_2) induction step

N5. $\quad S \subset \mathbb{N} \wedge 1 \in S \wedge (n \in S \Rightarrow n+1 \in S) \Rightarrow S = \mathbb{N}$

Suppose that (I_1) and (I_2) hold. Define

$$S := \{n \in \mathbb{N} : P_n \text{ is true}\}$$

$$\left.\begin{array}{l} (I_1) \Rightarrow 1 \in S \\ (I_2) \Rightarrow (n \in S \Rightarrow n+1 \in S) \end{array}\right| \overset{N5}{\Longrightarrow} S = \mathbb{N} \Leftrightarrow \begin{array}{l} \text{all statements } P_1, P_2, \dots \\ \text{are true} \end{array}$$

Example

Prove that for real $x > -1$ and for any $n \in \mathbb{N}$

$$(1+x)^n \geq 1 + nx$$

Solution: Fix $x > -1$. Denote P_n: "$(1+x)^n \geq 1+nx$".

- P_1: $1+x \geq 1+x$ is true (basis of induction)
- Suppose that P_n is true, i.e., $(1+x)^n \geq 1+nx$. Then
  $(1+x)^{n+1} = (1+x)^n(1+x) \geq (1+nx)(1+x) = 1+nx+x+nx^2 \geq 1+(n+1)x$, i.e.
  $P_n \Rightarrow P_{n+1}$. Induction step holds. By the principle of mathematical induction, P_n is true for all $n \in \mathbb{N}$.

Remark

Principle of mathematical induction with different basis

Let $P_1, P_2, P_3, \dots$ be a list of statements that may or may not be true. Let $k \in \mathbb{N}$. Then

(I_1') P_k is true

(I_2') P_n is true $\Rightarrow$ P_{n+1} is true for all $n \geq k$

$\Rightarrow$ all statements $P_k, P_{k+1}, P_{k+2}, \dots$ are true

Proof. Define $P_n' = P_{n+k-1}$, $n \in \mathbb{N}$, and apply the principle of mathematical induction for $P_1', P_2', P_3', \dots$ ■

Example Prove that for all $n \in \mathbb{N}$, $n \geq 2$ $\quad 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} > \sqrt{n}$

Solution. P_1: $1 > \sqrt{1}$ is false. P_2: $1 + \frac{1}{\sqrt{2}} > \sqrt{2}$ is true. For $n \geq 2$ if $1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} > \sqrt{n}$, then $1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} > \sqrt{n} + \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n^2+n}+1}{\sqrt{n+1}} > \sqrt{n+1}$

induction step is true. Principle of math. induction implies the result.

Integer and rational numbers

$\mathbb{Z} := \mathbb{N} \cup \{-n : n \in \mathbb{N}\} \cup \{0\} = \{0, 1, -1, 2, -2, \dots\}$ integer numbers

$\frac{1}{2} \notin \mathbb{Z}$

$\mathbb{Q} := \{ \frac{m}{n} : (m, n \in \mathbb{Z}) \wedge (n \neq 0) \}$ rational numbers

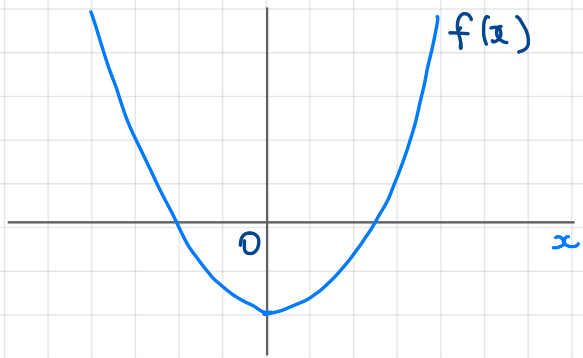
$\mathbb{Q} \setminus \{0\}$ is closed with respect to four arithmetic operations

Are there any other numbers?

Consider polynomial equation $x^2 = 2$

$(\pm\sqrt{2})^2 - 2 = 0$ $\sqrt{2} \notin \mathbb{Q}$

$f(x) = x^2 - 2$



Algebraic numbers

Definition 2.1 (Algebraic number)

A number is called algebraic if it satisfies a polynomial equation $c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0$, where $c_0, \ldots, c_n$ are integers and $n \geq 1$.

Remark Rational numbers are algebraic numbers: for $q = \frac{k}{l}$ take $n=1$, $c_0 = -k$, $c_1 = l$, giving the equation $lx - k = 0$

Examples of algebraic numbers:

$\sqrt[4]{17}$ $\qquad x^4 = 17$

$\sqrt{2}$ $\qquad x^2 = 2$

$\sqrt{2+\sqrt{2+\sqrt{2}}}$ $\qquad ((x^2-2)^2-2)^2-2 = x^8 - 8x^6 + 20x^4 - 16x^2 + 2 = 0$

$\sqrt{2} \notin \mathbb{Q}$

Theorem 2.2 (Rational Zeros Theorem)

Suppose that $c_0, c_1, \dots, c_n$ are integers and r is a rational number satisfying the polynomial equation

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0 \qquad (*)$$

Let $r = \frac{c}{d}$ where c and d are integers having no common factors. Then c divides c_0 and d divides c_n.

$\left(\frac{c}{d} \text{ solves } (*) \Rightarrow c \text{ divides } c_0,\ d \text{ divides } c_n \right)$

Proof. No proof. ∎

<u>Corollary</u>. If r satisfies $r^2 - 2 = 0$, then $r \notin \mathbb{Q}$.

<u>Proof</u>. Let r be such that $r^2 - 2 = 0$. If $r \in \mathbb{Q}$, then by Thm 2.2. $r \in \{1, 2, -1, -2\}$. If $r \in \{1, 2, -1, -2\}$, then $r^2 - 2 \in \{-1, 2\}$, $r^2 - 2 \neq 0$. Contradiction, $r \notin \mathbb{Q}$ ∎