MATH 142A: Introduction to Analysis

www.math.ucsd.edu/~ynemish/teaching/142a

Today: Natural, rational, algebraic numbers > Q&A: January 5
Next: Ross § 3

Week 1:

- visit course website
- homework 0 (due Friday, January 7)
- join Piazza

Logical symbolism

Common logical connectives

Example:

A: Alice plays accordion

B: Bob reads a book

C: Alice and Bob stay at home

Typical mathematical statement:

Typical proof:

Logical symbolism

Basic rules for constructing proofs

- if A is true and $A \Rightarrow B$,
- the law of excluded middle:
  ↳ used in proofs by contradiction
- rule of double negation:

Use words instead of symbols (most of the time)

$A \Rightarrow B$	$A \Leftrightarrow B$
A implies B	A is equivalent to B
B follows from A	A if and only if B
B is necessary condition for A	A is necessary and sufficient for B
A is sufficient condition for B	

Logical symbolism

Think about the following statements

Set theory notation

A set is a "collection of distinguishable objects"

- a set may consist of any distinguishable objects
- a set is uniquely determined by the collection of objects it consists of
- a set can be defined as a collection of objects having certain property
    - listing objects
    - the set of all objects x that satisfy property P

If S is a set, means that x is an element of S

x is not an element of S

Set theory notation

S, T are two sets, then means that

each element of T belongs to S.

Defining a set from another set by specifying a rule

Operations on sets

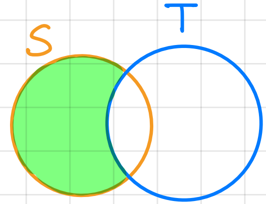
If we have 2 sets S, T, then

- $S \setminus T =$ is the difference between S and T
- $S \cup T =$ is the union of S and T
- $S \cap T =$ is the intersection of S and T

Set theory notation

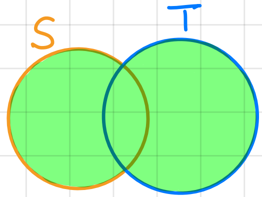
$\mathcal{A}$ is a set, S_α, $\alpha \in \mathcal{A}$, is a collection of sets, then

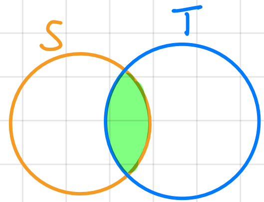
$$\bigcup_{\alpha \in \mathcal{A}} S_\alpha = \{x : x \in S_\alpha \text{ for at least one } \alpha \in \mathcal{A}\}$$



$$\bigcap_{\alpha \in \mathcal{A}} S_\alpha = \{x : x \in S_\alpha \text{ for all } \alpha \in \mathcal{A}\}$$

Examples

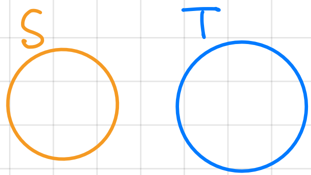




Empty set is the set with no elements, $\emptyset$

$S = \{1, 2, 3\}$

$T = \{4, 5, 6\}$



Natural numbers

We assume that we know what natural numbers are: numbers we use to count objects.

Peano Axioms:

N1. $1 \in \mathbb{N}$

N2. $n \in \mathbb{N} \Rightarrow n+1 \in \mathbb{N}$

N3. For any $n \in \mathbb{N}$, $n+1=1$ is false

N4. $(m, n \in \mathbb{N}) \wedge (m+1=n+1) \Rightarrow m=n$

N5.

Properties N1 – N5 define $\mathbb{N}$ uniquely.

Principle of mathematical induction

Let $P_1, P_2, P_3, \ldots$ be a list of statements that may or may not be true. Then

(I_1)

(I_2) $\Big| \Rightarrow$

(I_1) basis of induction $\quad$ (I_2) induction step

N5. $S \subset \mathbb{N} \wedge 1 \in S \wedge (n \in S \Rightarrow n+1 \in S) \Rightarrow S = \mathbb{N}$

Suppose that (I_1) and (I_2) hold. Define

$(I_1) \Rightarrow$

$(I_2) \Rightarrow$ $\Big| \Longrightarrow$ $\quad$ $\Leftrightarrow$

Example

Prove that for real $x > -1$ and for any $n \in \mathbb{N}$

$$(1+x)^n \geq 1 + nx$$

Solution: Fix $x > -1$. Denote P_n: $(1+x)^n \geq 1 + nx$.

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Remark

Principle of mathematical induction with different basis

Let $P_1, P_2, P_3, \dots$ be a list of statements that may or may not be true. Let $k \in \mathbb{N}$. Then

(I_1') P_k is true

(I_2') P_n is true $\Rightarrow$ P_{n+1} is true for all $n \geq k$

$\Rightarrow$ all statements $P_k, P_{k+1}, P_{k+2}, \dots$ are true

Proof. Define , $n \in \mathbb{N}$, and apply the principle of mathematical induction for

Example Prove that for all $n \in \mathbb{N}$, $n \geq 2$

Solution.

Integer and rational numbers

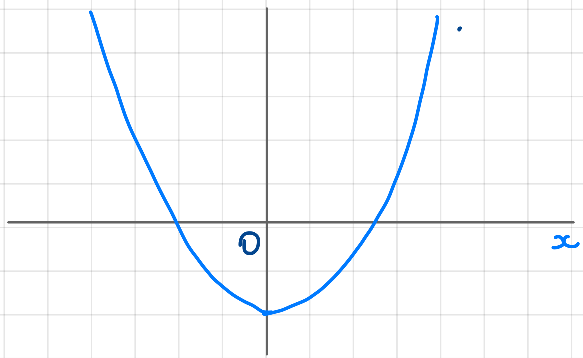
integer numbers

rational numbers

is closed with respect to four arithmetic operations

Are there any other numbers ?

Consider polynomial equation



Algebraic numbers

Definition 2.1 (Algebraic number)

A number is called algebraic if it satisfies a polynomial equation , where $c_0, \dots, c_n$ are integers and $n \geq 1$.

Remark Rational numbers are algebraic numbers: for $q = \frac{k}{\ell}$ take , giving the equation

Examples of algebraic numbers:

$\sqrt{2} \notin \mathbb{Q}$

Theorem 2.2 (Rational Zeros Theorem)

Suppose that $c_0, c_1, \dots, c_n$ are integers and r is a rational number satisfying the polynomial equation

Let $r = \frac{c}{d}$ where c and d are integers having no common factors.
Then c divides c_0 and d divides c_n.

Proof. No proof. ■

Corollary.

Proof.