☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem. Different parts of the same problem can be written on the same page (for example, part (a) and part (b))**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

☐ **From the moment you access the midterm problems on Gradescope you have 60 MINUTES to COMPLETE AND UPLOAD your exam to Gradescope. Plan your time accordingly.**

☐ **All steps of the proofs should be INCLUDED in your solutions. Provide references to the theorem/examples from the lectures/texbook used in your proofs.**

☐ **You are allowed to use the textbook, lecture notes and your personal notes. You are not allowed to use the electronic devices (except for accessing the online version of the textbook) or outside assistance. Outside assistance includes but is not limited to other people, the internet and unauthorized notes.**

1. (25 points) Prove that for all $n \in \mathbb{N}$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}. \tag{1}$$

Proof. We prove by induction on n. Let P_n be the statement that (1) holds for n.

For $n = 1$, LHS (the left hand side) of (1) is 1 and RHS of (1) is $2 - 1 = 1$. So P_1 holds.

If P_n is true, we shall prove P_{n+1}.

$$\begin{aligned} &1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2} \\ \leq\ & 2 - \frac{n^2 + 2n + 1 - n}{n(n+1)^2} = 2 - \frac{n^2 + n + 1}{n(n+1)^2} \\ <\ & 2 - \frac{n^2 + n}{n(n+1)^2} = 2 - \frac{1}{n+1}, \end{aligned}$$

where we used P_n on the first line. So P_{n+1} is true.

By mathematical induction, (1) holds for all $n \in \mathbb{N}$. $\square$

2. (25 points) Prove that
$$\lim_{n\to\infty} \sqrt[n]{n^2+n} = 1$$
using only the definition of convergence (i.e., without using any theorems about limits of sequences or examples from lectures/textbook). Clearly indicate how you choose $N(\varepsilon)$ for any $\varepsilon > 0$.

Proof. First we observe that $\sqrt[n]{n^2+n} \geq 1$.
Write n^2+n as
$$n^2+n = n(n+1) = \sqrt{n}\sqrt{n}\sqrt{n+1}\sqrt{n+1}\cdot 1 \cdots 1.$$

Then by the AM-GM inequality (Lecture 6),
$$\sqrt[n]{n^2+n} \leq \frac{2\sqrt{n}+2\sqrt{n+1}+n-4}{n} \tag{2}$$
$$= 1 + \frac{2\sqrt{n}+2\sqrt{n+1}-4}{n} \tag{3}$$
$$< 1 + \frac{4\sqrt{n+1}}{n} \tag{4}$$
$$< 1 + \frac{8}{\sqrt{n}}, \tag{5}$$

where on the last step we used that $n+1 < 4n$ for all $n \in \mathbb{N}$. We conclude that for any $n \in \mathbb{N}$
$$\left|\sqrt[n]{n^2+n} - 1\right| \leq \frac{8}{\sqrt{n}}. \tag{6}$$

Fix $\varepsilon > 0$ and choose
$$N(\varepsilon) := \left\lfloor \left(\frac{8}{\varepsilon}\right)^2 \right\rfloor. \tag{7}$$

Then for any $n > N(\varepsilon)$
$$\left|\sqrt[n]{n^2+n} - 1\right| \leq \frac{8}{\sqrt{n}} \leq 8 \cdot \frac{\varepsilon}{8} = \varepsilon, \tag{8}$$

which by definition means that $\lim_{n\to\infty} \sqrt[n]{n^2+n} = 1$. $\square$

3. (25 points) Compute the limit

$$\lim_{n\to\infty} \frac{\sqrt[3]{n^2}\sin(n!)}{n+1}.$$

Clearly indicate all the statements from the lectures/textbook used to compute the limit.

Proof. Recall that $|\sin(x)| \leq 1$ for any real number x. For $n \geq 10, n+1 \geq n/2$. So

$$\left|\frac{\sqrt[3]{n^2}\sin(n!)}{n+1}\right| \leq \frac{\sqrt[3]{n^2}}{n+1} \leq 2\frac{n^{\frac{2}{3}}}{n} \leq 2n^{-\frac{1}{3}}.$$

Since $\lim_{n\to\infty} 2n^{-\frac{1}{3}} = 0$, by Theorem 9.11-(ii) in Lecture 6,

$$\lim_{n\to\infty} \frac{\sqrt[3]{n^2}\sin(n!)}{n+1} = 0.$$

$\square$

4. (25 points) Prove that the sequence $(x_n)_{n=1}^{\infty}$ with

$$x_n = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{4}\right)\cdots\left(1 - \frac{1}{2^n}\right)$$

converges.

Proof. Clearly, $x_n \geq 0$.

$$x_{n+1} = x_n\left(1 - \frac{1}{2^{n+1}}\right) < x_n.$$

So (x_n) is a decreasing sequence and bounded below by 0. By Theorem 10.2, (x_n) converges. $\square$