☐ Write your name and PID on the top of EVERY PAGE.

☐ Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem. Different parts of the same problem can be written on the same page (for example, part (a) and part (b))

☐ Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.

☐ From the moment you access the midterm problems on Gradescope you have 65 MINUTES to COMPLETE AND UPLOAD your exam to Gradescope. Plan your time accordingly.

☐ All steps of the proofs should be INCLUDED in your solutions. Provide references to the theorem/examples from the lectures/texbook used in your proofs.

☐ You are allowed to use the textbook, lecture notes and your personal notes. You are not allowed to use the electronic devices (except for accessing the online version of the textbook) or outside assistance. Outside assistance includes but is not limited to other people, the internet and unauthorized notes.

1. (20 points) Numbers that are *not algebraic* are called *transcendental*.

   Show that a square of a transcendent number is a transcendental number.

   [**Hint:** Use proof by contradiction.]

   Solution. Let $b \in \mathbb{R}$, b is not algebraic.

   Proof by contradiction. Suppose that b^2 is algebraic. Then, by definition, b^2 is a solution to a certain polynomial equation with integer coefficients, i.e., there exist $n \in \mathbb{N}$ and $c_0, c_1, \ldots, c_n \in \mathbb{Z}$ such that

   $$c_n(b^2)^n + c_{n-1}(b^2)^{n-1} + \cdots + c_1(b^2) + c_0 = 0. \tag{1}$$

   Then (1) is a polynomial equation for b with integer coefficients. For a formal proof, define

   $$\tilde{c}_{2k} = c_k, \quad \tilde{c}_{2k+1} = 0, \qquad k \in \{0, 1, \ldots, n\}, \tag{2}$$

   then

   $$\tilde{c}_{2n} b^{2n} + \cdots + \tilde{c}_1 b^1 + \tilde{c}_0 = 0. \tag{3}$$

   This means that b is algebraic, which contradicts to the initial assumption that b is transcedental. We conclude that b^2 is not algebraic.

2. (20 points) Prove that

   $$\lim_{n\to\infty} \frac{2n}{n^2+1} = 0$$

   using only the definition of convergence (i.e., without using any theorems from lectures/textbook). Clearly indicate how you choose $N(\varepsilon)$ for any $\varepsilon > 0$.

   Solution. Fix $\varepsilon > 0$. Note that

   $$\left|\frac{2n}{n^2+1} - 0\right| = \frac{2n}{n^2+1} < \frac{2n}{n^2} = \frac{2}{n} \tag{4}$$

   and

   $$n > \frac{2}{\varepsilon} \Leftrightarrow \frac{2}{n} < \varepsilon, \tag{5}$$

   Take $N(\varepsilon) := \left[\frac{2}{\varepsilon}\right]$. Then for any $n > N(\varepsilon)$

   $$\left|\frac{2n}{n^2+1} - 0\right| = \frac{2n}{n^2+1} < \frac{2n}{n^2} = \frac{2}{n} < \varepsilon, \tag{6}$$

   which by definition means that

   $$\lim_{n\to\infty} \frac{2n}{n^2+1} = 0. \tag{7}$$

3. (20 points) Compute the limit

$$\lim_{n\to\infty} \frac{(n+1)(n+2)(n+3)}{n^3+n^2+1}.$$

Clearly indicate all the statements from the lectures/textbook used to compute the limit.

Solution. First, pull out the leading terms in the numerator and the denominator

$$\frac{(n+1)(n+2)(n+3)}{n^3+n^2+1} = \frac{n^3(1+\frac{1}{n})(1+\frac{2}{n})(1+\frac{3}{n})}{n^3(1+\frac{1}{n}+\frac{1}{n^3})} = \frac{(1+\frac{1}{n})(1+\frac{2}{n})(1+\frac{3}{n})}{1+\frac{1}{n}+\frac{1}{n^3}}. \tag{8}$$

By the example from Lecture 4 (or Important Example 1 from Lecture 7) and Theorem 9.2 (Lecture 5)

$$\lim_{n\to\infty} \frac{1}{n} = \lim_{n\to\infty} \frac{2}{n} = \lim_{n\to\infty} \frac{3}{n} = \lim_{n\to\infty} \frac{1}{n^3} = 0. \tag{9}$$

By Theorems 9.3, 9.4 (Lecture 5), limit of a sum is a sum of limit, limit of a product is a product of limits, therefore

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right) = (1+0)(1+0)(1+0) = 1, \tag{10}$$

$$\lim_{n\to\infty} \left(1+\frac{1}{n}+\frac{1}{n^3}\right) = 1+0+0 = 1. \tag{11}$$

By Theorem 9.6 (Lecture 5), limit of a fraction is a fraction of limits (note that the denominator converges to $1 \neq 0$) therefore

$$\lim_{n\to\infty} \frac{(1+\frac{1}{n})(1+\frac{2}{n})(1+\frac{3}{n})}{1+\frac{1}{n}+\frac{1}{n^3}} = \frac{1}{1} = 1. \tag{12}$$

4. (20 points) Compute the limit

$$\lim_{n\to\infty} \sqrt[n]{n+|\sin(n)|}.$$

Clearly indicate all the statements from the lectures/textbook used to compute the limit.

Solution. First, note that for any $n \in \mathbb{N}$

$$0 \le |\sin(n)| \le 1 \le n. \tag{13}$$

Therefore for any $n \in \mathbb{N}$

$$1 \le \sqrt[n]{n+|\sin(n)|} \le \sqrt[n]{n+n} = \sqrt[n]{2n} = \sqrt[n]{2}\cdot\sqrt[n]{n}. \tag{14}$$

By the important examples 3 and 4 from Lecture 6, $\lim_{n\to\infty} \sqrt[n]{2} = 1$, $\lim_{n\to\infty} \sqrt[n]{n} = 1$, therefore, by Theorem 9.4,

$$\lim_{n\to\infty} \sqrt[n]{2} \cdot \sqrt[n]{n} = 1. \tag{15}$$

Now (14) and (15) together with the queeze lemma imply that

$$\lim_{n\to\infty} \sqrt[n]{n + |\sin(n)|} = 1. \tag{16}$$

5. (20 points) Prove that the sequence $(x_n)_{n=1}^{\infty}$ with

$$x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

converges.

Solution. Sequence (x_n) is increasing: for any $n \in \mathbb{N}$

$$x_{n+1} - x_n = \frac{1}{(n+1)!} > 0 \quad \Rightarrow \quad x_{n+1} > x_n. \tag{17}$$

Also, sequence (x_n) is bounded: for any $n \in \mathbb{N}$

$$\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \le \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} = \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} < 2, \tag{18}$$

therefore, for any $n \in \mathbb{N}$

$$x_n < 3. \tag{19}$$

Sequence (x_n) is thus increasing and bounded above, therefore, by Theorem 10.2 (Lecture 7) sequence (x_n) converges.