MATH 142A - INTRODUCTION TO ANALYSIS
PRACTICE FINAL

WINTER 2022

1. Let $a, b, c \in \mathbb{R}$ be such that $a < b < c$ and $(c-a)(c-b) = (b-a)^2$. Show that

$$r := \frac{c-a}{b-a} \tag{1}$$

is not a rational number.

Hint: Show that r satisfies a polynomial equation with integer coefficients.

Solution. Since

$$r = \frac{c-a}{b-a}, \tag{2}$$

we have that

$$c - a = r(b-a) \quad \text{and} \quad c - b = (c-a) - (b-a) = (r-1)(b-a). \tag{3}$$

Plugging the above expressions into the equation $(c-a)(c-b) = (b-a)^2$ we get

$$(b-a)^2(r-1)r = (b-a)^2. \tag{4}$$

Since $b - a > 0$, the above equation implies that r satisfies the equation

$$r^2 - r - 1 = 0. \tag{5}$$

By Corollary 2.3, if r is a rational number, then $r \in \{-1, 1\}$. Neither $r = 1$ nor $r = -1$ satisfies Equation (5), therefore we conclude that r is not a rational number. (Number r is called the golden ratio)

2. Using only Definition 9.8 prove that

$$\lim_{n\to\infty} \log_{10}(\log_{10} n) = +\infty. \tag{6}$$

Clearly indicate how you chose $N(M)$ for any $M > 0$, and write explicitly $N(2)$, $N(5)$, $N(10)$.

Solution. Fix $M > 0$. Then for any $n > \lfloor 10^{10^M} \rfloor$

$$\log_{10}(\log_{10} n) > \log_{10}(\log_{10} 10^{10^M}) = M. \tag{7}$$

Therefore, by Definition 9.8

$$\lim_{n\to+\infty} \log_{10}(\log_{10} n) = +\infty \tag{8}$$

with $N(M) = \lfloor 10^{10^M} \rfloor$. In particular, $N(2) = 10^{100}$, $N(5) = 10^{100000}$, $N(10) = 10^{10^10}$. (This sequence converges to infinity very slowly)

3. Determine if the series

$$\sum_{n=1}^{\infty} \frac{2^n n!}{n^n} \tag{9}$$

converges. Justify your answer.

Solution. Denote

$$a_n := \frac{2^n n!}{n^n}. \tag{10}$$

Notice that

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n n!} = \frac{2n^n}{(n+1)^n} = \frac{2}{(1+\frac{1}{n})^n}. \tag{11}$$

By the Important Example from Lecture 7,

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e. \tag{12}$$

By Theorem 9.6,

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \frac{2}{\lim_{n\to\infty}(1+\frac{1}{n})^n} = \frac{2}{e}. \tag{13}$$

By the Important Example 16, $e > 2$, so $2/e < 1$. By Theorem 14.8 (Ratio test) we conclude that the series $\sum a_n$ converges.

4. Let $a \in \mathbb{R}$ and let $f : [a, +\infty) \to \mathbb{R}$ be a function such that

(i) $f \in C([a, +\infty))$
(ii) $\lim_{x\to+\infty} f(x) = p \in \mathbb{R}$

Prove that f is *uniformly continuous* on $[a, +\infty)$.

Solution. Fix $\varepsilon > 0$.

Since $\lim_{x\to+\infty} f(x) = p$, by the $\varepsilon - \delta$ definition of the limit (Lecture 18) there exists $M > a$ such that for any $x \in (M, +\infty)$

$$|f(x) - p| < \frac{\varepsilon}{2}. \tag{14}$$

Function f is continuous on $[a, M+1] \subset [a, +\infty)$, therefore by the Cantor-Heine Theorem (Theorem 19.2) f is uniformly continuous on $[a, M+1]$. By definition, this means that there exists $\delta > 0$ such that for all $x, y \in [a, M+1]$

$$|f(x) - f(y)| < \frac{\varepsilon}{2}. \tag{15}$$

Now for any $x, y \in [a, +\infty)$, $x < y$, $|x - y| < \min\{\delta, 1\}$, we have

- if $y \leq M + 1$, then by (15) $|f(x) - f(y)| < \varepsilon$.
- if $y > M + 1$, then $x > M$ and by (14) and the triangle inequality

$$|f(x) - f(y)| \leq |f(x) - p| + |f(y) - p| < \varepsilon. \tag{16}$$

We conclude that $x, y \in [a, +\infty)$ and $|x - y| < \min\{\delta, 1\}$ implies $|f(x) - f(y)| < \varepsilon$. By Definition (Lecture 15) this means that f is uniformly continuous on $[a, +\infty)$.

5. Compute the derivative of the function $f : (0, +\infty) \to \mathbb{R}$ given by

$$f(x) = x + x^x. \tag{17}$$

Provide all intermediate steps.

Solution. First, compute the derivative on x^x. For this, rewrite this function as

$$x^x = e^{\log x^x} = e^{x \log x}. \tag{18}$$

Function $x \log x$ is differentiable on $(0, +\infty)$, function e^x is differentiable on $\mathbb{R}$, therefore by Theorem 28.4 (about the derivative of a composition)

$$(x^x)' = \Big(e^{x\log x}\Big)' = e^{x\log x}\Big(x\log x\Big)' = e^{x\log x}\Big(\log x + 1\Big) = x^x\Big(\log x + 1\Big). \tag{19}$$

Therefore,

$$f'(x) = 1 + x^x\Big(\log x + 1\Big). \tag{20}$$

6. Prove that the inequality

$$py^{p-1}(x-y) \le x^p - y^p \le px^{p-1}(x-y) \tag{21}$$

holds for $0 < y < x$ and $p > 1$.

Solution. Consider function $f(x) = x^p$. Then for any interval $[y, x] \subset (0, +\infty)$, f is continuous on $[y, x]$ and differentiable on (y, x). Therefore, we can apply Lagrange's Mean Value Theorem (Theorem 29.3), which gives that there exists a number $\xi \in (y, x)$ such that

$$x^p - y^p = p\xi^{p-1}(x-y). \tag{22}$$

Since $p > 1$, $p - 1 > 0$, and $y < \xi < x$, we have that

$$y^{p-1} \le \xi^{p-1} \le x^{p-1}. \tag{23}$$

Together with (22) this implies that

$$py^{p-1}(x-y) \le x^p - y^p \le px^{p-1}(x-y). \tag{24}$$

7. Let

$$f : \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big) \to \mathbb{R}, \quad f(x) = \log(\cos x). \tag{25}$$

Find a polynomial $P(x)$ such that

$$f(x) - P(x) = o(x^3) \quad \text{as} \quad x \to 0. \tag{26}$$

Solution By the local Taylor's formula with the remainder in Peano's form, $P(x)$ is equal to the Taylor's polynomial of degree 3 at 0. In order to determine the coefficients of $P(x)$, compute the derivatives of f

$$f'(x) = (\log(\cos x))' = \frac{1}{\cos x}\cdot(-\sin x) = -\frac{\sin x}{\cos x}, \tag{27}$$

$$f''(x) = \Big(-\frac{\sin x}{\cos x}\Big)' = -\frac{\cos^2 x + \sin^2 x}{\cos^2 x} = -\frac{1}{\cos^2 x}, \tag{28}$$

$$f^{(3)}(x) = \Big(-\frac{1}{\cos^2 x}\Big)' = -2\frac{\sin x}{\cos^3 x}. \tag{29}$$

Now

$$f(0) = \log 1 = 0, \quad f'(0) = \tan 0 = 0, \quad f''(0) = -1, \quad f^{(3)}(0) = 0. \tag{30}$$

We conclude that

$$f(x) = -\frac{x^2}{2} + o(x^3) \quad \text{as} \quad x \to 0. \tag{31}$$