MATH180C: Introduction to Stochastic Processes II

Lecture A00: math-old.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math-old.ucsd.edu/~ynemish/teaching/180cB

Today: MC review. Conditioning on continuous random variables

Next: PK 7.1, Durrett 3.1

Week 4:

- homework 3 (due Saturday, April 23)
- Midterm 1: Friday, April 22

Example: Birth and death processes

If we consider the birth and death process, the equation $\pi Q = 0$ takes the following form

$$Q = \left(\begin{matrix} -\lambda_0 & \overset{= q_{01}}{\lambda_0} & \\ \underset{q_{10} =}{\mu_1} & -(\lambda_1+\mu_1) & \lambda_1 \\ & \mu_2 & \end{matrix} \right.$$

$$\left.\begin{aligned} &-\pi_0\lambda_0 + \pi_1\mu_1 = 0 \\ &\pi_0\lambda_0 - \pi_1(\lambda_1+\mu_1) + \pi_2\mu_2 = 0 \\ &\quad\vdots \\ &\pi_{i-1}\lambda_{i-1} - \pi_i(\lambda_i+\mu_i) + \pi_{i+1}\mu_{i+1} = 0 \end{aligned}\right| \Rightarrow \begin{aligned} &\pi_1 = \frac{\lambda_0}{\mu_1}\pi_0 \\ &\pi_2 = \frac{\lambda_1}{\mu_2}\pi_1 = \frac{\lambda_1\lambda_0}{\mu_2\mu_1}\pi_0 \\ &\pi_{i+1} = \frac{\lambda_i}{\mu_{i+1}}\pi_i = \cdots = \frac{\lambda_i\cdots\lambda_0}{\underbrace{\mu_{i+1}\cdots\mu_1}_{\theta_{i+1}}}\pi_0 \end{aligned}$$

where $\theta_i = \frac{\lambda_{i-1}}{\mu_i} \cdot \frac{\lambda_{i-2}}{\mu_{i-1}} \cdot \ldots \cdot \frac{\lambda_0}{\mu_1}$, $\theta_0 = 1$.

Then, $\sum_{i=0}^{\infty} \pi_i = 1$ implies that $\pi_0 \sum_{i=0}^{\infty} \theta_i = 1$

If $\sum_{i=0}^{\infty} \theta_i < \infty$, then (X_t) is positive recurrent and $\pi_j = \frac{\theta_j}{\sum_{i=0}^{\infty} \theta_i}$

If $\sum_{i=0}^{\infty} \theta_i = \infty$, then $\pi_j = 0 \ \forall j$.

Example. Linear growth with immigration

Birth and death process, $\lambda_j = \lambda j + a$, $\mu_j = \mu j$ $(*)$

Using Kolmogorov's equations we showed (lecture 5) that $E(X_t) \to \frac{a}{\mu-\lambda}$, $t \to \infty$, if $\mu > \lambda$.

What is the limiting distribution of X_t?

From the previous slide, $\pi_j = \frac{\theta_j}{\sum_{i=0}^{\infty} \theta_i}$, $\theta_j = \frac{\lambda_{j-1} \cdots \lambda_0}{\mu_j \cdots \mu_1}$

If we replace λ_j, μ_j by $(*)$, we get

$$\pi_j = \left(\frac{\lambda}{\mu}\right)^j \left(1 - \frac{\lambda}{\mu}\right)^{a/\lambda} \frac{\frac{a}{\lambda}\left(\frac{a}{\lambda}+1\right)\cdots\left(\frac{a}{\lambda}+j-1\right)}{j!}, \quad j > 1$$

$$\pi_0 = \left(1 - \frac{\lambda}{\mu}\right)^{-\frac{a}{\lambda}}$$

What you should know for midterm 1 (minimum):

- definition of continuous time MC, Markov property, transition probabilities, generator
- representations of MC: infinitesimal (generator), jump-and-hold, transition probabilities, rate diagram and relations between them (in particular Q and $P(t)$)
- computing absorption probabilities and mean time to absorption
- computing stationary distributions for finite and infinite state MCs and interpretation of $(\pi_i)_{i=0}^{\infty}$
- basic properties of birth and death processes

Conditioning on continuous r.v.

Def. Let X and Y be jointly distributed continuous random variables with joint probability density function $f_{X,Y}(x,y)$. We call the function

$$f_{X|Y}(x|y) := \frac{f_{X,Y}(x,y)}{f_Y(y)} \quad \text{for} \quad f_Y(y) > 0$$

the conditional probability density function of X given $Y=y$.

The function $$F_{X|Y}(x|y) = \int_{-\infty}^{x} f_{X|Y}(s|y)\,ds$$

is called conditional CDF of X given $Y=y$

Conditional expectation

Def. Let X and Y be jointly distributed continuous random variables, let $f_{X|Y}(x|y)$ be a conditional distribution of X given $Y=y$ and let $g: \mathbb{R} \to \mathbb{R}$ be a function for which $E(|g(X)|) < \infty$. Then we call

$$E(g(X) \mid Y=y) := \int_{-\infty}^{+\infty} g(x) f_{X|Y}(x|y)\, dx \quad \text{if } f_Y(y) > 0$$

the conditional expectation of $g(X)$ given $Y=y$.

In particular, if $g(x) = \mathbb{1}_A(x)$ indicator of set A, then

$$E(\mathbb{1}_A(X) \mid Y=y) = P(X \in A \mid Y=y) = \int_A f_{X|Y}(x|y)\, dx$$

Remark

If Y is a continuous random variable, then

$$P(Y=y)=0 \quad \text{for all } y\in\mathbb{R}$$

Therefore, we cannot define $P(X\in A \mid Y=y)$ as

$$P(X\in A \mid Y=y) = \xcancel{\frac{P(X\in A, Y=y)}{P(Y=y)}}$$

On the other hand consider example:

X, Y i,i.d. $X, Y \sim \text{Unif}[0,1]$, $Z = X - Y$

If $Y=\frac{1}{2}$, $Z = X - \frac{1}{2} \sim \text{Unif}[-\frac{1}{2}, \frac{1}{2}]$

makes perfect sense

Intuitive explanation / derivation

$$P(X\in[x,x+\Delta x],Y\in[y,y+\Delta y])$$
$$= f_{X,Y}(x,y)\,\Delta x\,\Delta y + o(\Delta x\cdot\Delta y)$$

Using the multiplication rule ($f_Y(y)>0$ on $[y,y+\Delta y]$)

$$P(X\in[x,x+\Delta x],Y\in[y,y+\Delta y])$$
$$= P(X\in[x,x+\Delta x]\mid Y\in[y,y+\Delta y])\,P(Y\in[y,y+\Delta y])$$

$$\frac{P(X\in[x,x+\Delta x]\mid Y\in[y,y+\Delta y])}{\Delta x} = \frac{P(X\in[x,x+\Delta x],Y\in[y,y+\Delta y])}{\frac{P(Y\in[y,y+\Delta y])}{\Delta y}\,\Delta x\,\Delta y}$$

$$\Big\downarrow\ \Delta x\to 0$$

$$\text{“}\ f_X(x\mid Y\in[y,y+\Delta y])\text{”}$$

$$\Big\downarrow\ \Delta y\to 0$$

$$\text{“}\ f_X(x\mid Y=y)\text{”} = f_{X|Y}(x\mid y)$$

$$\Big\downarrow\ \begin{matrix}\Delta x\to 0\\ \Delta y\to 0\end{matrix}$$

$$\frac{f_{X,Y}(x,y)}{f_Y(y)}$$