MATH180C: Introduction to Stochastic Processes II

Lecture A00: math.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math.ucsd.edu/~ynemish/teaching/180cB

Today: Asymptotic behavior of renewal processes

Next: PK 7.5, Durrett 3.1, 3.3

Week 7:

- homework 6 (due Monday, May 16, week 8)

Key renewal theorem

Suppose $H(t)$ is an unknown function that satisfies

$$H(t) = h(t) + H * F(t) \quad (*)$$

renewal equation

E.g.: $M(t) = F(t) + M * F(t)$,

$m(t) = f(t) + m * F(t) = f(t) + m * f(t)$

Remark about notation

- Convolution with c.d.f.: $g * F(t) = \int_{-\infty}^{+\infty} g(t-x)\, dF(x)$
- Convolution with p.d.f.: $g * f(t) = \int_{-\infty}^{+\infty} g(t-x) f(x)\, dx$

Def. Function h is called locally bounded if $\max_{0 \le x \le t} |h(x)| < \infty \; \forall t$

Def. Function h is absolutely integrable if

$$\int_0^\infty |h(x)|\, dx < \infty$$

Key renewal theorem

Thm (Key renewal theorem) Let h be locally bounded.

(a) If H satisfies $H = h + h * M$, then H is locally bounded and

$$H = h + H * F \quad (*)$$

(b) Conversely, if H is a locally bounded solution to $(*)$, then

$$H = h + h * M \quad (**)$$ [convolution in the Riemann–Stieltjes sense]

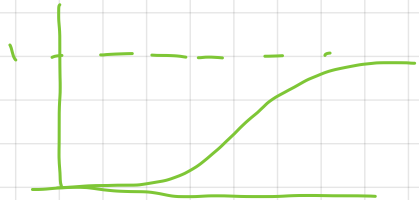
(c) If h is absolutely integrable, then

$$\lim_{t \to \infty} H(t) = \frac{\int_0^\infty h(x)\,dx}{\mu}$$

No proof.

Remark. Key renewal theorem says that if h is locally bounded, then there exists a unique locally bounded solution to $(*)$ given by $(**)$

Examples



- Renewal function: $M(t)$ satisfies

  and $M = F + F * M = F + M * F$

  $F(t)$ is nondecreasing, so (c) does not apply to the renewal equation for $M(t)$

- Renewal density: $m(t)$ satisfies

  and $m = f + f * m$

  $= f + f * M$ (in the Riemann - Stieltjes sense)

  f is absolutely integrable, $\int_0^\infty f(x)\,dx = 1$, so

$$\lim_{t \to \infty} m(t) = \frac{\int_0^\infty f(x)\,dx}{\mu} = \frac{1}{\mu}$$

Important remark

Let $W = (W_1, W_2, \dots)$ be arrival times of a renewal process, and denote $W' = (W_1', W_2', \dots)$ with

$$W_i' = W_{i+1} - W_1 = X_2 + X_3 + \dots + X_{i+1},$$

shifted arrival times.

Then:

- W' is independent of $W_1 = X_1$
- W' has the same distribution as W

Example

Example. Compute $\lim_{t\to\infty} E(\gamma_t)$. Take $H(t) = E(\gamma_t)$

If $X_1 > t$, then $\gamma_t = X_1 - t$; if $X_1 < t$ condition on $X_1 = s$

$$E(\gamma_t) = E\left((X_1 - t)\mathbb{1}_{X_1 > t}\right) + E\left(\gamma_t \mathbb{1}_{X_1 \le t}\right)$$

$$E(\gamma_t \mathbb{1}_{X_1 \le t}) = \int_0^\infty P\left((W_{N(t)+1} - t)\mathbb{1}_{X_1 \le t} > w\right) dw$$

$$= \int_0^\infty \sum_{k=1}^\infty P\left((W_k - t)\mathbb{1}_{X_1 \le t} > w, N(t) = k-1\right) dw$$

$$= \int_0^\infty \sum_{k=2}^\infty P\left(\left(X_1 + \sum_{j=2}^k X_j - t\right)\mathbb{1}_{X_1 \le t} > w, N(t) = k-1\right) dw$$

$$= \int_0^\infty \left[\sum_{k=2}^\infty \int_0^t P\left(\underbrace{\sum_{j=2}^k X_j}_{= W'_{k-1}} - (t-s) > w, \underbrace{N(t) = k-1}_{= N'(t-s) = k-2}\right) dF(s)\right] dw$$

$$= \int_0^t \left[\underbrace{\int_0^\infty \sum_{\ell=1}^\infty P\left(W'_\ell - (t-s) > w, N'(t-s) = \ell - 1\right) dw}_{P(\gamma'_{t-s} > w)}\right] dF(s) = \overset{H * F}{\overset{\shortparallel}{\int_0^t E(\gamma_{t-s})\, dF(s)}}$$

Example (cont)

Assume that $E(X_1)=\mu$, $Var(X_1)=\sigma^2$

$$E\left((X_1-t)\mathbb{1}_{X_1>t}\right) = \int_t^\infty (x-t)\,dF(x) = \int_t^\infty (t-x)\,d(1-F(x))$$

$$= (t-x)(1-F(x))\Big|_t^\infty + \int_t^\infty (1-F(x))\,dx$$

0

Since we assume that $Var(X_1)=\sigma^2$,

and Ex: $x(1-F(x)) \to 0$ as $t \to \infty$

Finally, we have that

$$H(t) = \int_t^\infty (1-F(x))\,dx + H * F(t)$$

therefore $H(t) = h(t) + h * M(t)$

with $h(t) = \int_t^\infty (1-F(x))\,dx$