MATH180C: Introduction to Stochastic Processes II

Lecture A00: math.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math.ucsd.edu/~ynemish/teaching/180cB

Today: Asymptotic behavior of renewal processes

Next: PK 7.5, Durrett 3.1, 3.3

Week 7:

- homework 6 (due Monday, May 16, week 8)

Midterm 2: Wednesday, May 18

Key renewal theorem

Thm (Key renewal theorem) Let h be locally bounded.

(a) If H satisfies $H = h + h * M$, then H is locally bounded and

$$H = h + H * F \quad (*)$$

(b) Conversely, if H is a locally bounded solution to $(*)$, then

$$H = h + h * M \quad (**)$$

[convolution in the Riemann-Stieltjes sense]

(c) If h is absolutely integrable, then

$$\lim_{t \to \infty} H(t) = \frac{\int_0^\infty h(x)\,dx}{\mu}$$

Example. $H(t) = E(\gamma_t)$

Last time:

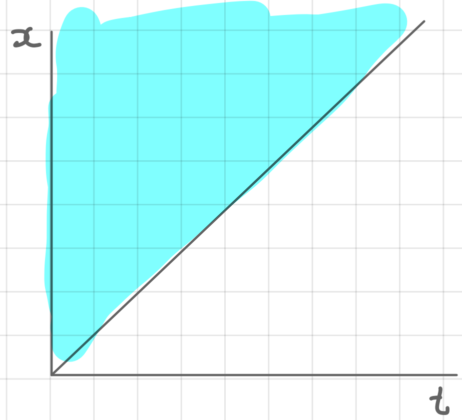
$$H(t) = \int_t^\infty (1 - F(x))\,dx + H * F(t)$$

$$H(t) = h(t) + h * M(t) \quad \text{with} \quad h(t) = \int_t^\infty (1 - F(x))\,dx$$

Example (cont)

In particular,

$$\int_0^\infty \overbrace{\int_t^\infty (1-F(x))\,dx}^{h(t)}\,dt = \int_0^\infty \left(\int_0^x (1-F(x))\,dt \right) dx$$



$$= \int_0^\infty \overbrace{(1-F(x))}^{P(X_1>x)}\, x\,dx = \frac{1}{2} E[X_1^2]$$

$$= \frac{1}{2}(\sigma^2+\mu^2) < \infty \quad \Rightarrow h(t) \text{ is (absolutely) integrable}$$

$\Rightarrow$ by part (c) of the key renewal theorem

$$\lim_{t\to\infty} E(\gamma_t) = \frac{\sigma^2+\mu^2}{2\cdot\mu}$$

Similarly $\displaystyle \lim_{t\to\infty} E(\delta_t) = \frac{\sigma^2+\mu^2}{2\mu}$, $\displaystyle \lim_{t\to\infty} E(\beta_t) = \frac{\sigma^2+\mu^2}{\mu} > \mu$

Example

What is the expected time to the next earthquake in the long run?

For $X_1 \sim \text{Unif}[0,1]$

$$E(X_1^2) = \int_0^1 x^2\,dx = \frac{1}{3} = \sigma^2 + \mu^2$$

therefore, $\lim_{t\to\infty} E(\gamma_t) = \dfrac{\sigma^2+\mu^2}{2\mu} = \dfrac{\frac{1}{3}}{2\cdot\frac{1}{2}} = \dfrac{1}{3}$

And the long run expected time between two consecutive earthquakes is $\frac{2}{3} > \frac{1}{2} = E(X_1)$

Remark: moments of nonnegative r.v.s

Proposition. Let X be a nonnegative random variable. Then

$$E(X^n) = n\int_0^\infty x^{n-1} P(X > x)\,dx$$

$$= n\int_0^\infty x^{n-1} (1 - F(x))\,dx$$

$n=1$: $E(X) = \int_0^\infty (1-F(x))dx$

$n=2$: $E(X^2) = 2\int_0^\infty x(1-F(x))dx$

Proof. $X \geq 0 \Rightarrow X^n \geq 0$. Using the "tail" formula for the expectation of nonnegative random variables

$$E(X^n) = \int_0^\infty P(X^n > t)\,dt = \int_0^\infty P(X > t^{1/n})\,dt$$

After the change of variable $x = t^{1/n}$ we get

$$E(X^n) = n\int_0^\infty x^{n-1} P(X > x)\,dx = n\int_0^\infty x^{n-1} (1 - F(x))\,dx$$

$\blacksquare$

Remark. $M(t)$ is finite for all t

Proposition. Let $N(t)$ be a renewal process with interrenewal times X_i having distribution F. If there exist $c>0$ and $\alpha\in(0,1)$ such that $P(X_1>c)>\alpha$, then $M(t)=E(N(t))<\infty \quad \forall\, t$

Proof: Recall that $M(t)=\sum_{k=1}^{\infty} P(W_k\le t)=\sum_{k=1}^{\infty} P\left(\sum_{j=1}^{k} X_j\le t\right)$ $(*)$

Fix $t>0$, $L\in\mathbb{N}$ such that $c\cdot L>t$. Then

$$P\left(\sum_{j=1}^{L} X_j>t\right)\ge P(X_1>c,\, X_2>c,\dots,X_L>c)>\alpha^L>0$$

$$P\left(\sum_{j=1}^{L} X_j\le t\right)\le 1-\alpha^L<1.$$ Thus, for any $n\in\mathbb{N}$

$$P(W_{nL}\le t)=P\left(\sum_{j=1}^{nL} X_j\le t\right)\le (1-\alpha^L)^n,$$ from which we conclude (exercise) that $\sum_{k=1}^{\infty} P(W_k\le t)=M(t)<\infty$ ∎

Example: Age replacement policies (PK, p. 363)

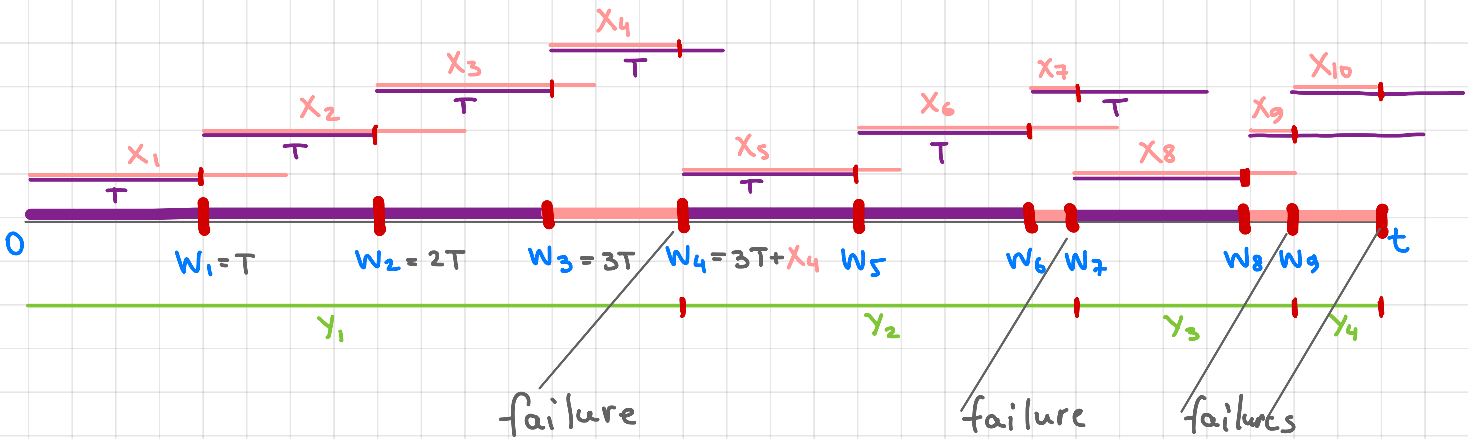
Setting:
- component's lifetime has distribution function F
- component is replaced
  (A) either when it fails,
  (B) or after reaching age T (fixed)
  whichever occurs first
- replacements (A) and (B) have different costs:
  replacement of a failed component (A) is more expensive than the planned replacement (B)

Question: How does the long-run cost of replacement depend on the cost of (A), (B) and age T?
What is the optimal T that minimizes the long-run cost of replacement?

Example: Age replacement policies (PK, p. 363)

Notation: X_i - lifetime of i-th component, $F_{X_i}(t) = F(t)$

Y_i - times between failures



Here we have two renewal processes

(1) renewal process $N(t)$ generated by renewal times $(W_i)_{i=1}^{\infty}$

(2) renewal process $Q(t)$ generated by interrenewal times $(Y_i)_{i=1}^{\infty}$

$N(t) = \#$ replacements on $[0,t]$, $Q(t) = \#$ failure replacements on $[0,t]$

Example: Age replacement policies (PK, p. 363)

Compute the distribution of the interrenewal times for $N(t)$

$$W_i - W_{i-1} = \begin{cases} X_i, & \text{if } X_i \leq T \\ T, & \text{if } X_i > T \end{cases}, \text{ so}$$

$$F_T(x) := P(W_i - W_{i-1} \leq x) = \begin{cases} F(x), & x < T \\ 1, & x \geq T \end{cases}$$

In particular,

$$E(W_i - W_{i-1}) = \int_0^T (1 - F(x))\,dx =: \mu_T \leq \mu = E(X_i)$$

Using the elementary renewal theorem for $N(t)$, the total number of replacements has a long-run rate

$$\frac{E(N(t))}{t} \approx \frac{1}{\mu_T} \quad \text{for large } t$$

Example: Age replacement policies (PK, p. 363)

Compute the distribution of the interrenewal times for $\textcircled{1}(t)$.

$$Y_1 = \begin{cases} X_1 & \text{if } X_1 \le T \\ T + X_2 & \text{if } X_1 > T,\ X_2 \le T \\ \vdots & \\ Tn + X_{n+1}, & \text{if } X_1 > T, \dots, X_n > T,\ X_{n+1} \le T \\ \vdots & \end{cases}$$

so $\quad Y_1 = L \cdot T + Z$, where $P(L \ge n) = (1 - F(T))^n$, $Z \in [0,T]$

and for $z \in [0,T]$

$$P(Z \le z) = P(X_1 \le z,\ X_1 \cancel{\le} T) + P(X_2 \le z,\ X_1 > T,\ X_2 \cancel{\le} T)$$
$$+ \cdots + P(X_{n+1} \le z,\ X_1 > T, \dots, X_n > T,\ X_{n+1} \cancel{\le T}) + \cdots$$

$$= P(X_1 \le z) + P(X_2 \le z)\, P(X_1 > T) + \cdots + P(X_{n+1} \le z,\ X_1 > T, \dots, X_n > T)$$

$$= F(z)\left(1 + (1 - F(T)) + \cdots + (1 - F(T))^n + \cdots\right) = \frac{F(z)}{F(T)}$$