MATH180C: Introduction to Stochastic Processes II

Lecture A00: math-old.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math-old.ucsd.edu/~ynemish/teaching/180cB

Today: Introduction. Birth processes

Next: PK 6.2-6.3

Week 1:

- visit course web site
- homework 0 (due Friday April 1)
- join Piazza

Stochastic (random) processes

Def. Let $(\Omega, \mathcal{F}, P)$ be a probability space.
Stochastic process is a collection $(X_t : t \in T)$ of random variables $X_t : \Omega \to S \subset \mathbb{R}$ (all defined on the same probability space)

- often t represents time, but can be 1-D space
- T is called the index set, S is called the state space
- $X : \Omega \times T \to S$ $\quad (X_t(\omega) \in S)$
- for any fixed ω, we get a realization of all random variables $(X_t(\omega) : t \in T) \leftarrow$ sample path / trajectory

$$X_{\cdot}(\omega) : T \to S$$

- stochastic process is a random function

Stochastic processes. Classification

Questions:

- What is T
- What is S
- Relations between X_{t_1} and X_{t_2} for $t_1 \neq t_2$?
- Properties of the trajectory

Discrete time

$T = \mathbb{N}, \mathbb{Z},$ finite set

random vector

Continuous time

$T = \mathbb{R}, [0, +\infty), [0, 1]$

Real-valued

$S = \mathbb{R}$

Integer-valued

$S = \mathbb{Z}$

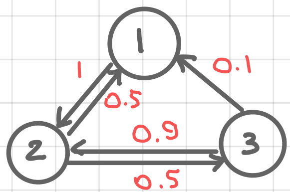
Nonnegative ...

$S \subset [0, +\infty)$

Continuous, right-continuous (càdlàg) sample path

Examples of stochastic processes

- Gaussian processes: for any $t \in T$, X_t has normal distrib.
- Stationary processes: distribution doesn't change in time
- Processes with stationary and independent increments (Lévy)
- Poisson process: increments are independent and Poisson$(\cdot)$
- Markov processes: "distribution in the future depends only on the current state, but does not depend on the past"

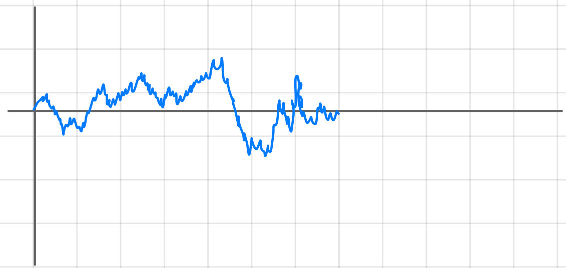


Examples of stochastic processes

- Martingales : $\mathbb{E}[X_{n+1} | X_n, \dots, X_1, X_0] = X_n$ ("fair game")

- Brownian motion (Wiener process) is a continuous-time st. proc.
  Gaussian, martingale, has stationary and independent increments, Markov, $\mathrm{Var}[W_t] = t$
  $\mathrm{Cov}[W_t, W_s] = \min\{s,t\}$, its sample path is everywhere continuous and nowhere differentiable

- diffusion processes (stochastic differential equations)



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Continuous time MC

Continuous Time Markov Chains

Def (Discrete-time Markov chain)

Let $(X_n)_{n\geq 0}$ be a discrete time stochastic process taking values in $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ (for convenience). $(X_n)_{n\geq 0}$ is called Markov chain if for any $n \in \mathbb{N}$ and $i_0, i_1, \dots, i_{n-1}, i, j \in \mathbb{Z}_+$

$$P(X_{n+1} = j \mid X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}, X_n = i) = P(X_{n+1} = j \mid X_n = i)$$

Def (Continuous-time Markov chain)

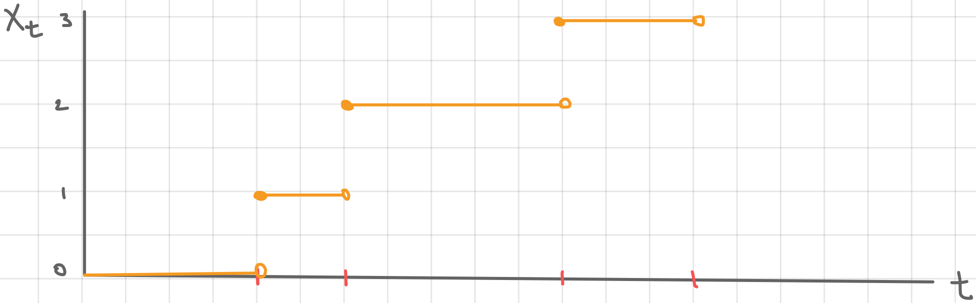
Let $(X_t)_{t\geq 0} = (X_t : 0 \leq t < \infty)$ be a continuous time process taking values in $\mathbb{Z}_+$. $(X_t)_{t\geq 0}$ is called Markov chain if for any $n \in \mathbb{N}$, $0 \leq t_0 < t_1 < \dots < t_{n-1} < s$, $t > 0$, $i_0, i_1, \dots, i_{n-1}, i, j \in \mathbb{Z}_+$

$$\mathbb{P}[X_{s+t} = j \mid X_{t_0} = i_0, \dots, X_{t_{n-1}} = i_{n-1}, X_s = i] \overset{(*)}{=} \mathbb{P}[X_{s+t} = j \mid X_s = i]$$

Poisson process

Def. A Poisson process of intensity (rate) $\lambda > 0$ is an integer-valued stochastic process $(X_t)_{t \geq 0}$ for which

1) for each time points $t_0 = 0 < t_1 < \cdots < t_n$, the process increments $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent random variables
2) for $s \geq 0$ and $t > 0$, the random variable $X_{s+t} - X_s$ has the Poisson distribution $\mathbb{P}[X_{t+s} - X_s = k] = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$, $k = 0, 1, \ldots$
3) $X_0 = 0$



Example: Poisson process as MC

Is Poisson process a continuous time MC?

Poisson process:

- √ continuous time
- √ discrete state
- √ (*)

Let $(X_t)_{t\geq 0}$ be a Poisson process, let $i_0 \leq i_1 \leq \dots \leq i_{n-1} \leq i \leq j$

$$P(X_{s+t} = j \mid X_{t_0} = i_0, X_{t_1} = i_1, \dots, X_{t_{n-1}} = i_{n-1}, X_s = i)$$

$$= \frac{\mathbb{P}[X_{t_0} = i_0, X_{t_1} - X_{t_0} = i_1 - i_0, \dots, X_s - X_{t_{n-1}} = i - i_{n-1}, X_{t+s} - X_s = j - i]}{\mathbb{P}[X_{t_0} = i_0, X_{t_1} - X_{t_0} = i_1 - i_0, \dots, X_s - X_{t_{n-1}} = i - i_{n-1}]}$$

$$= \mathbb{P}[X_{t+s} - X_s = j - i]$$

$$\mathbb{P}[X_{t+s} = j \mid X_s = i] = \frac{\mathbb{P}[X_s = i, X_{t+s} - X_s = j - i]}{\mathbb{P}[X_s = i]} = \mathbb{P}[X_{t+s} - X_s = j - i]$$

Transition probability function

One way of describing a continuous time MC is by using the transition probability functions.

Def. Let $(X_t)_{t\geq 0}$ be a MC. We call

$$\mathbb{P}[X_{s+t} = j \mid X_s = i] \qquad i,j \in \{0,1,\dots\}, \quad s \geq 0, \quad t > 0$$

the transition probability function for $(X_t)_{t\geq 0}$.

If $P(X_{s+t} = j \mid X_s = i)$ does not depend on s, we say that $(X_t)_{t\geq 0}$ has stationary transition probabilities and we define

$$P_{ij}(t) := \mathbb{P}[X_{t+s} = j \mid X_s = i] = \mathbb{P}[X_t = j \mid X_0 = i]$$

[compare with n-step transition probabilities]