MATH180C: Introduction to Stochastic Processes II

Today: Introduction. Birth processes

Next: PK 6.2-6.3

Week 1:

- visit course web site
- homework 0 (due Friday April 1)
- join Piazza

Stochastic (random) processes

<u>Def.</u> Let $(\Omega, \mathcal{F}, P)$ be a probability space.
Stochastic process is a collection $(X_t : t \in T)$
of random variables   (all defined on the
same probability space)

- often t represents time, but can be 1-D space
- T is called the index set, S is called the state space
- $X : \Omega \times T \to S$ ($X_t(\omega) \in S$)
- for any fixed ω, we get a realization of all
  random variables $(X_t(\omega) : t \in T) \leftarrow$ sample path trajectory

- stochastic process

Stochastic processes. Classification

Questions:

- What is T
- What is S
- Relations between X_{t_1} and X_{t_2} for $t_1 \neq t_2$?
- Properties of the trajectory

Discrete time   Continuous time

$T = \mathbb{N}, \mathbb{Z}$, finite set $T = \mathbb{R}, [0, +\infty), [0, 1]$
    ↳ random vector

Real-valued   Integer-valued   Nonnegative ...

$S = \mathbb{R}$ $S = \mathbb{Z}$ $S \subset [0, +\infty)$

Continuous, right-continuous (cadlag) sample path

Examples of stochastic processes

- Gaussian processes: for any $t \in T$, X_t has normal distrib.

- Stationary processes: distribution doesn't change in time

- Processes with stationary and independent increments (Levy

- Poisson process: increments are independent and Poisson($\cdot$)

- Markov processes: "distribution in the future depends only on the current state, but does <u>not</u> depend on the past"
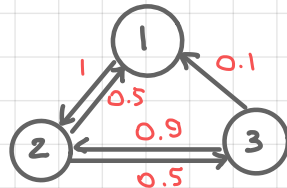
Examples of stochastic processes

- Martingales : $\mathbb{E}[X_{n+1} | X_n, ..., X_1, X_0] = X_n$ ("fair game")

- Brownian motion (Wiener process) is a continuous-time st. proc.
    Gaussian, martingale, has stationary and
    independent increments, Markov, $Var[W_t] = t$
    $Cov[W_t, W_s] = \min\{s, t\}$, its sample path is
    everywhere continuous and nowhere differentiable

- diffusion processes (stochastic
  differential equations)

- ...
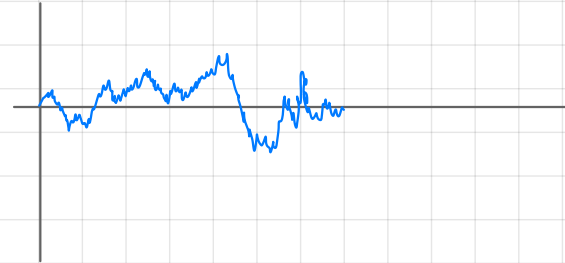
Continuous time MC

Continuous Time Markov Chains

Def (Discrete-time Markov chain)

Let $(X_n)_{n\geq 0}$ be a discrete time stochastic process taking values in $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ (for convenience). $(X_n)_{n\geq 0}$ is called Markov chain if for any $n \in \mathbb{N}$ and $i_0, i_1, \ldots, i_{n-1}, i, j \in \mathbb{Z}_+$

$$P(X_{n+1} = j \mid X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}, X_n = i) = P(X_{n+1} = j \mid X_n = i)$$

Def (Continuous-time Markov chain)

Let $(X_t)_{t\geq 0} = (X_t : 0 \leq t < \infty)$ be a continuous time process taking values in $\mathbb{Z}_+$. $(X_t)_{t\geq 0}$ is called Markov chain if for any $n \in \mathbb{N}$, $0 \leq t_0 < t_1 < \cdots < t_{n-1} < s$, $t > 0$, $i_0, i_1, \ldots, i_{n-1}, i, j \in \mathbb{Z}_+$
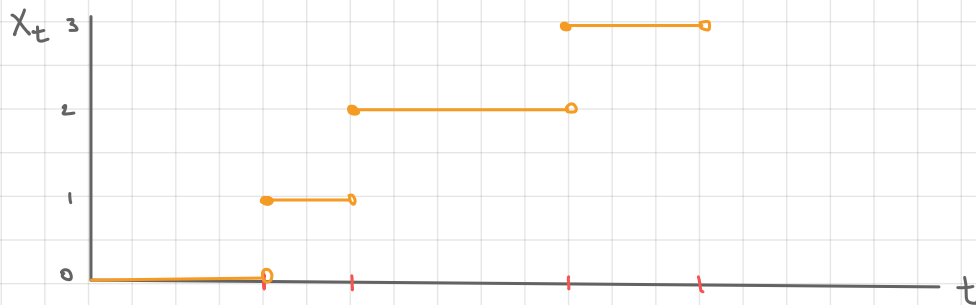
Poisson process

<u>Def</u>. A Poisson process of intensity (rate) $\lambda > 0$ is an integer-valued stochastic process $(X_t)_{t \geq 0}$ for which

1) for each time points $t_0 = 0 < t_1 < \cdots < t_n$, the process increments $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent random variables

2) for $s \geq 0$ and $t > 0$, the random variable $X_{s+t} - X_s$ has the Poisson distribution $\mathbb{P}[X_{t+s} - X_s = k] = \dfrac{(\lambda t)^k}{k!} e^{-\lambda t}$, $k = 0, 1, \ldots$

3) $X_0 = 0$

Example: Poisson process as MC

Is Poisson process a continuous time MC?
Poisson process:

    continuous time

    discrete state

    (*)

Let $(X_t)_{t \geq 0}$ be a Poisson process, let $i_0 \leq i_1 \leq \cdots \leq i_{n-1} \leq i \leq j$

$$P(X_{s+t} = j \mid X_{t_0} = i_0, X_{t_1} = i_1, \ldots, X_{t_{n-1}} = i_{n-1}, X_s = i)$$

$$=$$

$$=$$

$$=$$

Transition probability function

One way of describing a continuous time MC is by using the transition probability functions.

<u>Def</u>. Let $(X_t)_{t \geq 0}$ be a MC. We call

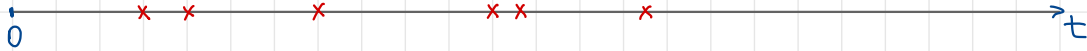$$i, j \in \{0, 1, \dots\}, \quad s \geq 0, \quad t > 0$$

the transition probability function for $(X_t)_{t \geq 0}$.

If $P(X_{s+t} = j \mid X_s = i)$ does not depend on s, we say that $(X_t)_{t \geq 0}$ has stationary transition probabilities and we define

[compare with n-step transition probabilities]

Characterization of the Poisson process

Experiment: count events occurring along $[0, +\infty)$ { time or 1-D space

Denote by $N((a,b])$ the number of events that occur on $(a,b]$.

Assumptions:

1. Number of events happening in disjoint intervals are independent.

2. For any $t \geq 0$ and $h > 0$, the distribution of $N((t,t+h])$ does not depend on t (only on h, the length of the interval)

3. There exists $\lambda > 0$ s.t. $P(N((t,t+h]) \geq 1) = \lambda h + o(h)$ as $h \to 0$
   (rare events)

4. Simultaneous events are not possible: $P(N((t,t+h]) \geq 2) = o(h), h \to 0$

Transition probabilities of the Poisson process

Let $(X_t)_{t \geq 0}$ be the Poisson process.

Define the transition probability functions

$$P_{ij}(h) := P(X_{t+h} = j \mid X_t = i), \quad i, j \in \{0, 1, 2, \ldots\}, \quad t \geq 0, \ h > 0$$

What are the infinitesimal (small h) transition probability functions for $(X_t)_{t \geq 0}$? As $h \to 0$,

$$P_{ii}(h) = P(X_{t+h} = i \mid X_t = i)$$

$$=$$

$$P_{i,i+1}(h) = P(X_{t+h} = i+1 \mid X_t = i) =$$

$$\sum_{j \notin \{i, i+1\}} P_{ij}(h) =$$

Poisson process and transition probabilities

To sum up: $(X_t)_{t \geq 0}$ is a MC with (infinitesimal) transition probabilities satisfying

$$P_{ii}(h) =$$

$$P_{i,i+1}(h) =$$

$$\sum_{j \notin \{i, i+1\}} P_{i,j}(h) =$$

What if we allow $P_{ij}(h)$ depend on i?

$\hookrightarrow$ birth and death processes

Pure birth processes

Def Let $(\lambda_k)_{k \geq 0}$ be a sequence of positive numbers. We define a pure birth process as a Markov process $(X_t)_{t \geq 0}$ whose stationary transition probabilities satisfy

1. $P_{k,k+1}(h) =$

2. $P_{k,k}(h) =$

3. $P_{k,j}(h) =$

4. $X_0 = 0$

Related model. Yule process : $\lambda_k = \beta k$ for some $\beta > 0$.

Describes the growth of a population

- birth rate is proportional to the size of the population

Birth processes and related differential equations

Now define $P_n(t) = P(X_t = n)$. For small $h > 0$

$$P_n(t+h) = P(X_{t+h} = n) =$$

$$=$$

$$=$$

$$=$$

$$=$$

$$P_n(t+h) - P_n(t) = -\lambda_n h P_n(t) + \lambda_{n-1} h P_{n-1}(t) + o(h)$$

$$P_n'(t) =$$

Birth processes and related differential equations

$P_n(t)$ satisfies the following system
of differentian eqs.   with initial conditions

$$(*) \begin{cases} P_0'(t) = \\[6pt] P_1'(t) = \\[6pt] P_2'(t) = \\[6pt] \vdots \\[6pt] P_n'(t) = \\[6pt] \vdots \end{cases}$$
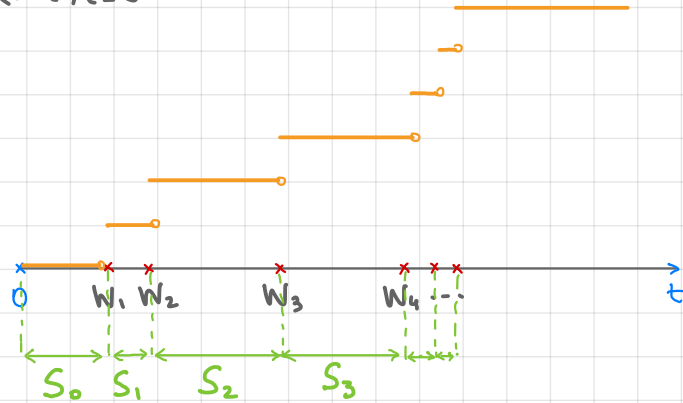
$P_0(0) =$

$P_1(0) =$

$P_2(0) =$

$\vdots$

$P_n(0) =$

$\vdots$

Solving this system gives the p.m.f. of X_t for any t

Description of the birth processes via sojourn times

$(X_t)_{t \geq 0}$

W_i – i-th "birth time" S_i – "time between $(i-1)$-th birth and i-th birth"

$$W_i = \sum_{\ell=0}^{i-1} S_\ell$$ $\hookrightarrow$ sojourn times

Alternative way of characterizing $(X_t)_{t \geq 0}$:

—

—

Description of the birth processes via sojourn times •

Theorem

Let $(\lambda_k)_{k \geq 0}$ be a sequence of positive numbers. Let $(X_t)_{t \geq 0}$ be a non-decreasing right-continuous process, $X_0 = 0$, taking values in $\{0, 1, 2 \dots\}$. Let $(S_i)_{i \geq 0}$ be the sojourn times associated with $(X_t)_{t \geq 0}$, and define $W_\ell = \sum_{i=0}^{\ell - 1} S_i$.
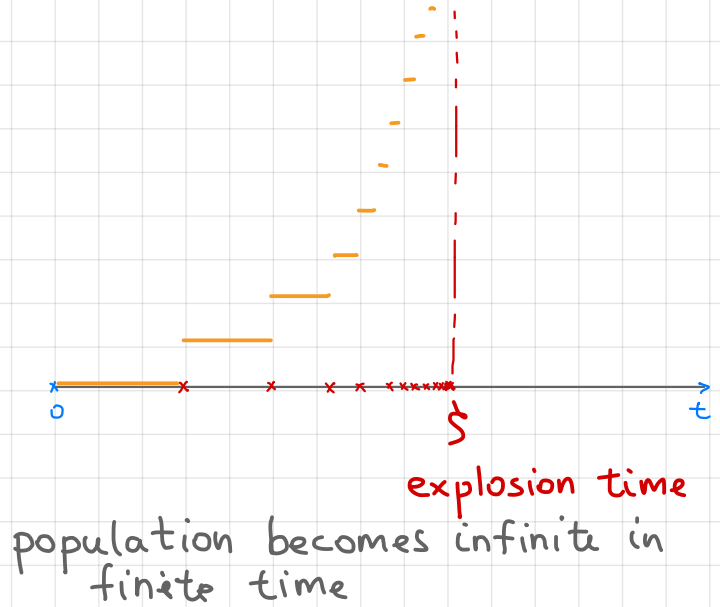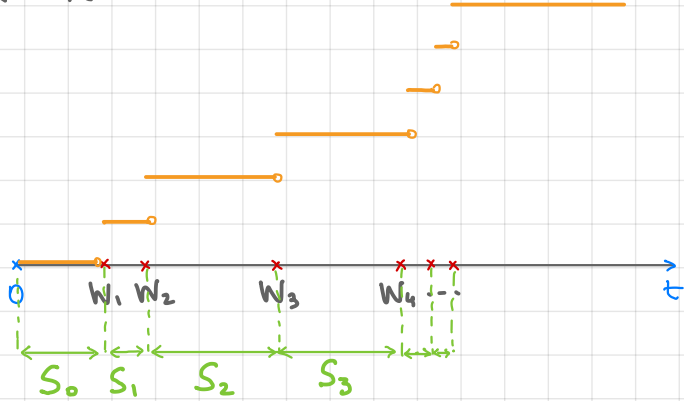
Then conditions

(a)

(b)

are equivalent to

(c)

Explosion

$(X_t)_{t \geq 0}$

S_0 S_1 S_2 S_3

explosion time

population becomes infinite in finite time

$\underline{\text{Thm}}$. Let $(X_t)_{t \geq 0}$ be a pure birth process of rates $(\lambda_k)_{k \geq 0}$. Then

Solving the system of differential equations (*)

$$(*) \begin{cases} P_0'(t) = -\lambda_0 P_0(t), & P_0(0) = 1 \\[2mm] P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t), & P_n(0) = 0 \quad \text{for} \quad n \geq 1 \end{cases}$$

$P_0(t):$

$$P_0'(t) =$$

$$\frac{P_0'(t)}{P_0(t)} =$$

$$g'(t) =$$

$$g(t) =$$

Solving the system of differential equations (*)

$P_n(t)$, $n \geq 1$

Consider the function $Q_n(t) =$

$(Q_n(t))' =$

$Q_n(t) =$

$\quad \hookrightarrow P_n(t) = \qquad\qquad\qquad\qquad \leftarrow$ apply recursively

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (if $\lambda_1 \neq \lambda_0$)

$$P_1(t) = e^{-\lambda_1 t} \int_0^t \lambda_0 e^{\lambda_1 s} e^{-\lambda_0 s} ds = e^{-\lambda_1 t} \int_0^t \lambda_0 e^{(\lambda_1 - \lambda_0)s} ds$$

$$= e^{-\lambda_1 t} \frac{\lambda_0}{\lambda_1 - \lambda_0} \left(e^{(\lambda_1 - \lambda_0)t} - 1 \right) = \frac{\lambda_0}{\lambda_1 - \lambda_0} e^{-\lambda_0 t} - \frac{\lambda_0}{\lambda_1 - \lambda_0} e^{-\lambda_1 t}$$