MATH180C: Introduction to Stochastic Processes II

Lecture A00: math.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math.ucsd.edu/~ynemish/teaching/180cB

Today: Brownian motion

Next: PK 8.1-8.2

Week 9:

- homework 7 (due Friday, May 27)
- HW6 regrades are active on Gradescope until May 28, 11 PM

Brownian motion. History

- Critical observation: Robert Brown (1827), botanist, movement of pollen grains in water
- First (?) mathematical analysis of Brownian motion: Louis Bachelier (1900), modeling stock market fluctuations
- Brownian motion in physics: Albert Einstein (1905) and Marian Smoluchowski (1906), explained the phenomenon observed by Brown
- First rigorous construction of mathematical Brownian motion: Norbert Wiener (1923)

Brownian motion $\underset{\text{in mathematics}}{\overset{}{=}}$ Wiener process

Brownian motion. Motivation

- almost all interesting classes of stochastic processes contain Brownian motion: BM is a
  - martingale
  - Markov process
  - Gaussian process
  - Lévy process (independent stationary increments)
- BM allows explicit calculations, which are impossible for more general objects
- BM can be used as a building block for other processes
- BM has many beautiful mathematical properties

Brownian motion. Definition

Def. **Brownian motion** with diffusion coefficient σ^2 is a continuous time stochastic process $(B_t)_{t\geq 0}$ satisfying

(i) $B(0)=0$, $B(t)$ is continuous as a function of t

(ii) For all $0\leq s<t<\infty$ $B(t)-B(s)$ is a Gaussian r.v. with mean 0 and variance $\sigma^2(t-s)$

(iii) The increments of B are independent: if $0\leq t_0<t_1<\dots<t_n$ then $\{B_{t_i}-B_{t_{i-1}}\}_{i=1}^{n}$ are independent Gaussian r.v.s

$\sigma^2=1 \leftarrow$ standard BM

BM as a continuous time continuous space Markov process

Recall: continuous time discrete space MC $(X_t)_{t\geq 0}$ is characterized by the transition probability function

$$P_{ij}(t) = P(X_{s+t} = j \mid X_s = i)$$

($(X_t)_{t\geq 0}$ has stationary transition probability functions)

In particular, $P(X_{s+t} \in A \mid X_s = i) = \sum_{j\in A} P_{ij}(t)$

In the continuous state space case the transition probabilities are described by the transition density

(i) $p_t(x,y) \geq ,\ \int_{-\infty}^{+\infty} p_t(x,y)\,dy = 1 \quad \forall\ t, x$

(ii) $P(X_{s+t} \in A \mid X_s = x) = \int_A p_t(x,y)\,dy$ for any $x \in \mathbb{R}$, $A \subset \mathbb{R}$

$\uparrow$ density of X_{s+t} given $X_s = x$

BM as a continuous time continuous space Markov process

Propotition. Let $(B_t)_{t\geq 0}$ be a standard BM. Then $(B_t)_{t\geq 0}$ is a Markov process with transition density

$$p_t(x,y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}(y-x)^2}$$

Informal explanation: Independent stationary increments imply that $(B_t)_{t\geq 0}$ is Markov with stationary transition density. Given $B_s = x$, $B_{s+t} = B_s + B_{s+t} - B_s \sim N(x,t)$ information before time s is irrelevant.

$$P(B_{s+t} \leq u \mid B_s = x) = P(B_s + (B_{s+t} - B_s) \leq u \mid B_s = x)$$

$$= P(x + B_{t+s} - B_s \leq u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} dy$$

BM as a continuous time continuous space Markov process

Let $t_1 < t_2 < \cdots < t_n < \infty$, $(a_i, b_i) \subset \mathbb{R}$. Then

$$P(B_{t_1} \in (a_1, b_1), B_{t_2} \in (a_2, b_2)) =$$

$$= \int_{-\infty}^{+\infty} P(B_{t_1} \in (a_1, b_1), B_{t_2} \in (a_2, b_2) \mid B_{t_1} = x_1)\, p_{t_1}(0, x_1)\, dx_1$$

$$= \int_{a_1}^{b_1} P(B_{t_2} \in (a_2, b_2) \mid B_{t_1} = x_1)\, p_{t_1}(0, x_1)\, dx_1$$

$$= \iint_{(a_1, b_1) \times (a_2, b_2)} p_{t_1}(0, x_1)\, p_{t_2 - t_1}(x_1, x_2)\, dx_1\, dx_2$$

More generally,

$$P(B_{t_1} \in (a_1, b_1), B_{t_2} \in (a_2, b_2), \dots, B_{t_n} \in (a_n, b_n))$$

$$= \int \cdots \int_{(a_1, b_1) \times \cdots \times (a_n, b_n)} p_{t_1}(0, x_1)\, p_{t_2 - t_1}(x_1, x_2) \cdots p_{t_n - t_{n-1}}(x_{n-1}, x_n)\, dx_1 \cdots dx_n$$

Diffusion equation. Transition semigroup. Generator

Let $(X_t)_{t\geq 0}$ be a Markov process.

Suppose we want to know how the distribution of X_t evolves in time:

$$E(f(X_{s+t}) \mid X_s = x) = \int_{-\infty}^{+\infty} f(y)\, p_t^X(x,y)\,dy =: P_t f(x)$$

CK

We call $(P_t)_{t\geq 0}$ the transition semigroup $\left[P_{s+t} f(x) = P_s(P_t f(x))\right]$

<u>Proposition</u> Let $(P_t)_{t\geq 0}$ be the transition semigroup of BM.

Then (i) the "infinitesimal generator" of $P(t)$ is given by

$$Q f(x) = \frac{1}{2} \frac{d^2}{dx^2} f(x)$$

(ii) density p_t satisfies $\frac{\partial}{\partial t} p_t(x,y) = \frac{1}{2} \frac{\partial^2}{\partial x^2} p_t(x,y)$ [K backward]

(iii) density p_t satisfies $\frac{\partial}{\partial t} p_t(x,y) = \frac{1}{2} \frac{\partial^2}{\partial y^2} p_t(x,y)$ [K forward]

diffusion equation

BM as a Gaussian process

<u>Def</u>. Stochastic process $(X_t)_{t\geq 0}$ is called a Gaussian process if for any $0 \leq t_1 < t_2 < \dots < t_n$

$(X_{t_1}, \dots, X_{t_n})$ is a Gaussian vector, or equivalently for any $c_1, \dots, c_n \in \mathbb{R}$

is a Gaussian r.v.

Recall that the distribution of a Gaussian vector is uniquelly defined by its mean and covariance matrix.

Similarly, each Gaussian process is uniquely described by

$$\mu(t) = E(X_t) \quad \text{and} \quad \Gamma(s,t) = \mathrm{Cov}(X_s, X_t) \geq 0$$

↑covariance function

BM as a Gaussian process

Proposition BM is a Gaussian process with

and

Proof. For any $0 \leq t_1 < t_2 < \cdots < t_n$, $B_{t_j} - B_{t_{j-1}}$ are indep. Gaussian, thus $\sum_{i=1}^{n} c_i B_{t_i} =$

is also Gaussian.

By definition . Let $s < t$.

Then $\Gamma(s,t) =$

$=$

$=$

$=$

Some properties of BM

Proposition. Let $(B_t)_{t\geq 0}$ be a standard BM. Then

(i) For any $s>0$, the process is a BM independent of $(B_u, 0 \leq u \leq s)$.

(ii) The process is a BM

(iii) For any $c>0$, the process is a BM

(iv) The process $(X_t)_{t\geq 0}$ defined by for $t>0$ is a BM.

Proof (i) Define $X_t = B_{t+s} - B_s$. Then

$\Rightarrow$ independent Gaussian increments,

$(X_t)_{t\geq 0}$ has continuous paths $\Rightarrow$

(iv) X_t is Gaussian, for $s<t$

Proof of $\lim_{t\to 0} X_t = 0$ is more technical, thus omitted.