MATH180C: Introduction to Stochastic Processes II

Lecture A00: math.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math.ucsd.edu/~ynemish/teaching/180cB

Today: Brownian motion

Next: PK 8.1-8.2

Week 10:

CAPES

- homework 8 (due Friday, June 3)
- HW7 regrades are active on Gradescope until June 4, 11 PM
- homework 9 and solutions are available on the course website

Reflection principle

Thm. Let $(B_t)_{t\geq 0}$ be a standard BM. Then for any $t \geq 0$ and $x > 0$

$$\cancel{(S_t)_{t\geq 0} \overset{(d)}{=} (|B_t|)_{t\geq 0}}$$

$$P(\underbrace{\max_{0\leq u\leq t} B_u}_{=: S_t} > x) = P(|B_t| > x)$$

Proof. Let $\tau_x = \min\{t : B_t = x\}$. Note that τ_x is a stopping time and is uniquely determined by $\{B_u, 0 \leq u \leq \tau_x\}$. From the definition of τ_x, $\max_{0\leq u\leq t} B_u \geq x \Leftrightarrow \tau_x \leq t$. Then

$$P(\max_{0\leq u\leq t} B_u \geq x, B_t < x) = P(\tau_x \leq t, B_{(t-\tau_x)+\tau_x} - \underbrace{B_{\tau_x}}_{=x} < 0)$$

$$\overset{SMP}{=} \frac{1}{2} P(\tau_x \leq t) = \frac{1}{2} P(\max_{0\leq u\leq t} B_u \geq x)$$

Now $P(\max_{0\leq u\leq t} B_u \geq x) = P(B_t \geq x) + P(\max_{0\leq u\leq t} B_u \geq x, B_t < x)$

$$\Rightarrow P(\max_{0\leq u\leq t} B_u \geq x) = 2P(B_t \geq x) = P(|B_t| \geq x) \blacksquare$$

Application of the RP: distribution of the hitting time τ_x

By definition, $\tau_x \le t \iff \max\limits_{0\le u\le t} B_u \ge x$, so

$$P(\tau_x \le t) = P\left(\max_{0\le u\le t} B_u \ge x\right) = 2P\left(B_t \ge x\right)$$

$$= 2\cdot\frac{1}{\sqrt{2\pi t}}\int_x^\infty e^{-\frac{u^2}{2t}}\,du \qquad \left\{ u = v\sqrt{t}\,,\ du = \sqrt{t}\,dv\right.$$

$$= \sqrt{\frac{2}{\pi t}}\int_{x/\sqrt{t}}^\infty e^{-\frac{v^2}{2}}\,dv$$

$\Rightarrow$ p.d.f. of τ_x $\quad f_{\tau_x}(t) = \sqrt{\frac{2}{\pi}}\,e^{-\frac{x^2}{2t}}\cdot\frac{1}{2}\,t^{-3/2} = \frac{1}{\sqrt{2\pi}}\,t^{-3/2}\,e^{-\frac{x^2}{2}}$

<u>Thm</u>. $F_{\tau_x}(t) = \sqrt{\frac{2}{\pi}}\int_{x/\sqrt{t}}^\infty e^{-\frac{v^2}{2}}\,dv$,

$$f_{\tau_x}(t) = \frac{x}{\sqrt{2\pi}}\,t^{-3/2}\,e^{-\frac{x^2}{2t}}$$

Zeros of BM

Denote by $\theta(t,t+s)$ the probability that $B_u=0$ on $(t,t+s)$

$$\theta(t,t+s) := P\big(B_u = 0 \text{ for some } u \in (t,t+s)\big)$$

Thm. For any $t,s>0$

$$\theta(t,t+s) = \frac{2}{\pi}\arccos\sqrt{\frac{t}{t+s}}$$

Proof Compute $P(B_u = 0 \text{ for some } u \in (t,t+s])$ by conditioning on the value of B_t.

$$\theta(t,t+s) = \int_{-\infty}^{+\infty} P\big(B_u=0 \text{ for some } u\in(t,t+s] \,\big|\, B_t = x\big) \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx \qquad (*)$$

Define $\tilde{B}_u = B_{t+u} - B_t$. Then

$$P(B_u = 0 \text{ on } (t,t+s] \,|\, B_t = x) = P(\tilde{B}_u = -x \text{ on } (0,s] \,|\, B_t = x)$$

$$\overset{MP}{=} P(\tilde{B}_u = -x \text{ on } (0,s]) = P(\tilde{B}_u = x \text{ on } (0,s]) \qquad (**)$$

Zeros of BM

Plugging (**) into (*) gives

$$\Theta(t, t+s) = \int_{-\infty}^{+\infty} P(B_u = x \text{ for some } u \in (0,s]) \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$$

$$= \int_{0}^{+\infty} P(B_u = x \text{ for some } u \in (0,s]) \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$$

$$+ \int_{0}^{\infty} P(B_u = -x \text{ for some } u \in (0,s]) \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$$

$$= \sqrt{\frac{2}{\pi t}} \int_{0}^{\infty} P(B_u = x \text{ for some } u \in (0,s]) e^{-\frac{x^2}{2t}} dx$$

Finally, $P(B_u = x > 0 \text{ for some } u \in (0,s]) = P(\max_{0 \le u \le s} B_u \ge x) = P(\tau_x \le s)$

$$(*) = \int_0^{\infty} \sqrt{\frac{2}{\pi t}} e^{-\frac{x^2}{2t}} \left(\int_0^s \frac{x}{\sqrt{2\pi}} y^{-3/2} e^{-\frac{x^2}{2y}} dy \right) dx = \frac{1}{\pi\sqrt{t}} \int_0^s \left(\int_0^{\infty} x e^{-\frac{x^2}{2}\left(\frac{1}{t} + \frac{1}{y}\right)} dx \right) y^{-3/2} dy$$

Zeros of BM

$$\int_0^\infty x e^{-\frac{x^2}{2}\left(\frac{1}{t}+\frac{1}{y}\right)} dx = \frac{1}{\frac{1}{t}+\frac{1}{y}} = \frac{ty}{t+y} \qquad w=\frac{x^2}{2}$$

$$\Rightarrow (*) = \frac{1}{\pi\sqrt{t}} \int_0^s \frac{ty}{t+y}\, y^{-3/2} dy = \frac{\sqrt{t}}{\pi} \int_0^s \frac{1}{(t+y)\sqrt{y}} dy$$

Now use the change of variable $z=\sqrt{\frac{y}{t}}$, $dy = 2t\,dz$

$$(*) = \frac{\sqrt{t}}{\pi} \int_0^{\sqrt{s/t}} \frac{1}{t(1+z^2)\sqrt{t}\,z} \cdot 2t\,dz = \frac{2}{\pi}\int_0^{\sqrt{s/t}} \frac{1}{1+z^2} dz = \frac{2}{\pi}\arctan\left(\sqrt{\frac{s}{t}}\right)$$

$$= \frac{2}{\pi}\arccos\left(\sqrt{\frac{t}{s+t}}\right)$$

exercise

$\blacksquare$

<u>Remark</u> Let $T_0 := \inf\{t>0 : B_t = 0\}$. Then $P(T_0=0)=1$

There is a sequense of zeros of $B_t(\omega)$ converging to 0.

To understand the structure of the set of zeros $\rightarrow$ Cantor set

Behavior of BM as $t \to \infty$

Thm. Let $(B_t)_{t\geq 0}$ be a (standard) BM. Then

$$P\left(\sup_{t\geq 0} B_t = +\infty , \inf_{t\geq 0} B_t = -\infty \right) = 1$$

(BM "oscilates with increasing amplitude")

Proof. Denote $Z = \sup_{t\geq 0} B_t$. Then for any $c > 0$

$$cZ = \sup_{t\geq 0} c \cdot B_t = \sup_{t \geq 0} c \cdot B_{\frac{t}{c^2}}$$

By property (iii), cB_{t/c^2} is a standard BM, so cZ has the same distribution as $Z \Rightarrow P(Z=0) = p$, $P(Z=\infty) = 1-p$

$$p = P(Z=0) \leq P\left(B_1 \leq 0 \text{ and } \sup_{t\geq 0} B_{t+1} - B_1 = 0 \right) = \frac{1}{2} \cdot P(Z=0) = \frac{1}{2} \cdot p$$

$\Rightarrow P(Z=0) = 0$, $P(Z=\infty) = 1$. Similarly for $\inf_{t\geq 0} B_t$ ∎

Sample paths of $(B_t)_t$ are not differentiable

Thm. $P(\ B_t$ is not differentiable at zero $)=1$

Proof. $P(\ \sup_{t\geq 0} B_t = \infty,\ \inf_{t\geq 0} B_t = -\infty)=1.$ $(\star)$

Consider $\tilde{B}_t = t B_{1/t}$. $(\tilde{B}_t)_{t\geq 0}$ is a BM (by property (iv))

By $(\star)$, for any $\varepsilon>0$ $\exists\ t<\varepsilon,\ s<\varepsilon$ such that

$\tilde{B}_t>0,\ \tilde{B}_s<0 \Rightarrow$ only differentiable if $\tilde{B}'_0=0$

But if $\tilde{B}'_0=0$, then

for some $t>0$ and all $0<s<t$,

which imples that for all $0<s<t$, which

contradicts to $(\star)$ ∎

Thm $P(\ (B_t)_{t\geq 0}$ is nowhere differentiable $)=1$

Reflected BM

Def. Let $(B_t)_{t\geq 0}$ be a standard BM. The stochastic process

$$R_t := |B_t| = \begin{cases} B_t, & \text{if } B(t) \geq 0 \\ -B_t, & \text{if } B(t) < 0 \end{cases}$$

is called reflected BM.

Think of a movement in the vicinity of a boundary.

Moments: $E(R_t) =$

$$Var(R_t) = E(B_t^2) - \left(E(|B_t|)\right)^2 =$$

Transition density: $P(R_t \leq y \mid R_0 = x) =$

$$=$$

$$\Rightarrow p_t(x,y) =$$

Thm (Lévy) Let $M_t = \max\limits_{0\leq u\leq t} B_u$. Then $(M_t - B_t)_{t\geq 0}$ is a reflected BM.

Reflected BM