MATH180C: Introduction to Stochastic Processes II

Lecture A00: math-old.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math-old.ucsd.edu/~ynemish/teaching/180cB

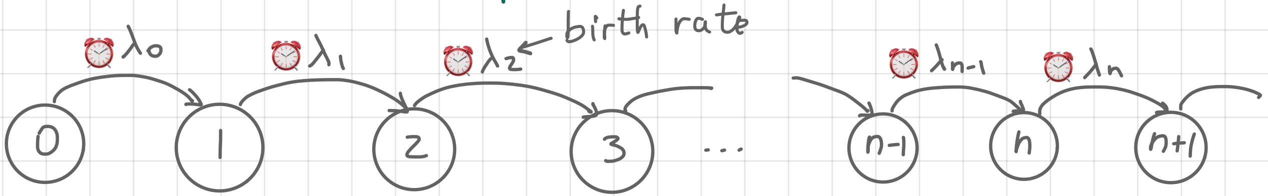
Today: Birth and death processes. Strong Markov property. Hitting probabilities

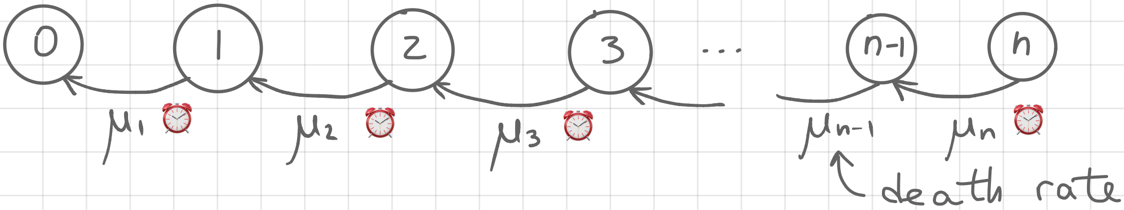
Next: PK 6.5, 6.6, Durrett 4.1

Week 2:

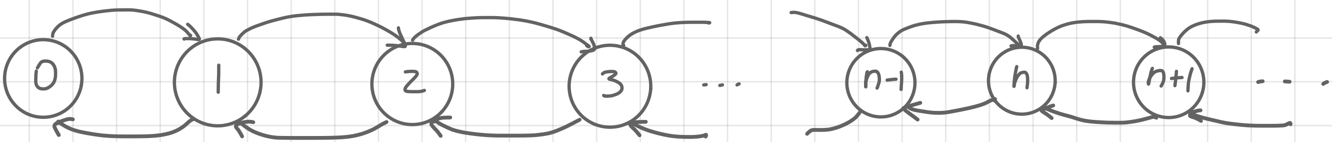
- homework 1 (due Friday April 8)

Birth and death processes





Combine both



Birth and death processes

Infinitesimal definition

Def. Let $(X_t)_{t\geq 0}$ be a continuous time MC, $X_t \in \{0,1,2,\dots\}$ with stationary transition probabilities. Then $(X_t)_{t\geq 0}$ is called a birth and death process with birth rates (λ_k) and death rates (μ_k) if

1. $P_{i,i+1}(h) = \lambda_i h + o(h)$
2. $P_{i,i-1}(h) = \mu_i h + o(h)$
3. $P_{i,i}(h) = 1 - (\lambda_i + \mu_i) h + o(h)$
4. $P_{ij}(0) = \delta_{ij}$ $\quad \left(P(X_0 = j \mid X_0 = i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \right)$
5. $\mu_0 = 0, \ \lambda_0 > 0, \ \lambda_i, \mu_i > 0$

Example: Linear growth with immigration

Dynamics of a certain population is described by the following principles:

during any small period of time of length h

- each individual gives birth to one new member with probability $\beta h + o(h)$ independently of other members;
- each individual dies with probability $\alpha h + o(h)$ independently of other members;
- one external member joins the population with probability $a h + o(h)$

Can be modeled as a Markov process

Example: Linear growth with immigration

Let $(X_t)_{t\geq 0}$ denote the size of the population.

Using a similar argument as for the Yule/pure death models:

pure birth growth

- $P_{n,n+1}(h) = n\beta h + ah + o(h)$

immigration growth

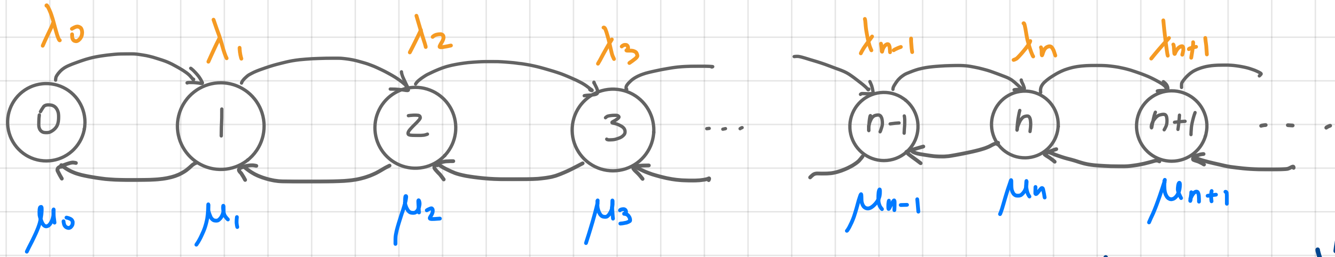
- $P_{n,n-1}(h) = n\alpha h + o(h)$
- $P_{n,n}(h) = 1 - (n\beta h + ah + n\alpha h) + o(h)$

↳ birth and death process with

$$\lambda_n = n\beta + a$$

$$\mu_n = n\alpha$$

Alternative (jump and hold) characterization



$\lambda = \mu = 1 \qquad \lambda' = \mu' = 2$

Sojourn times S_k are independent, $\frac{1}{2} = \frac{\lambda}{\lambda+\mu} = \frac{\mu}{\lambda+\mu} = \frac{\lambda'}{\lambda'+\mu'} = \frac{\mu'}{\lambda'+\mu'}$

$\mathrm{Exp}(2) \qquad \mathrm{Exp}(4)$

Each transition has two parts

- wait in state i for time $\sim \mathrm{Exp}(\lambda_i + \mu_i)$
- then choose where to go:

go $\rightarrow$ (i+1) with probability $\frac{\lambda_i}{\lambda_i + \mu_i}$

go (i-1) $\leftarrow$ with pobability $\frac{\mu_i}{\lambda_i + \mu_i}$

Stopping times

Def (Informal). Let $(X_t)_{t\geq 0}$ be a stochastic process and let $T \geq 0$ be a random variable. We call T a **stopping time** if the event

$$\{T \leq t\}$$

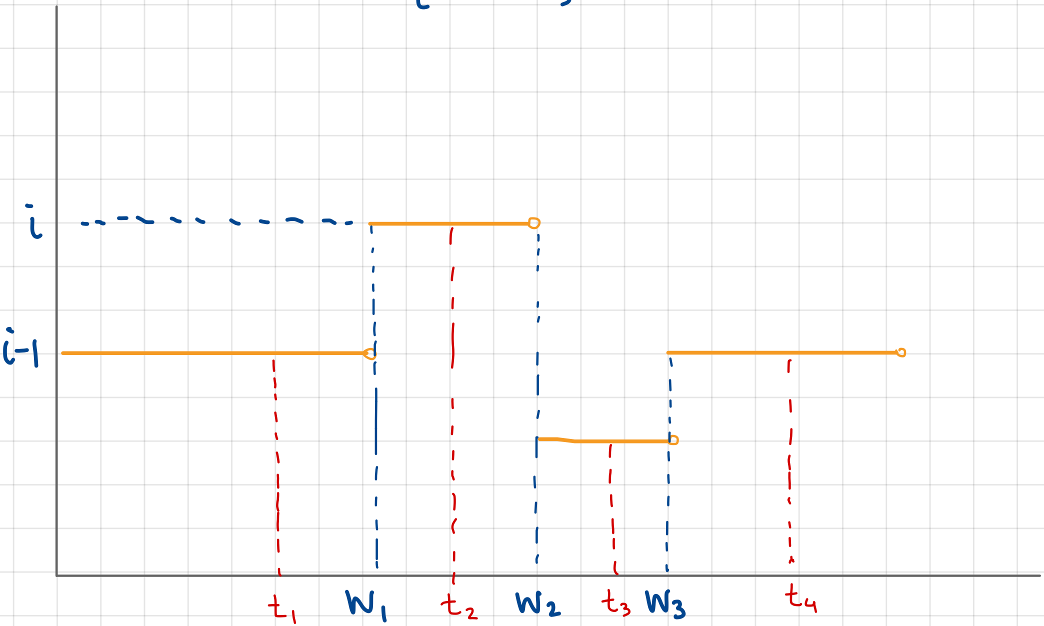
can be determined from the knowledge of the process up to time t (i.e., from $\{X_s : 0 \leq s \leq t\}$)

Examples: Let $(X_t)_{t\geq 0}$ be right-continuous

1. $\min\{t \geq 0 : X_t = i\}$ is a stopping time
2. W_k is a stopping time
3. $\sup\{t \geq 0 : X_t = i\}$ is not a stopping time

Stopping times

$$\{T \le t\}$$



Strong Markov property

Theorem (no proof)

Let $(X_t)_{t\geq 0}$ be a MC, let T be a stopping time of $(X_t)_{t\geq 0}$. Then, conditional on $T<\infty$ and $X_T=i$,

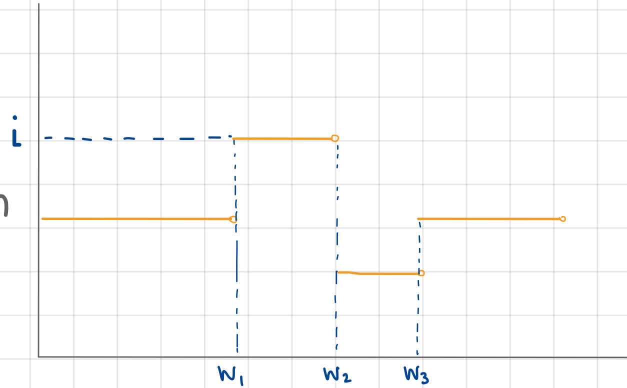
$$(X_{T+t})_{t\geq 0}$$

(i) is independent of $\{X_s,\ 0\leq s\leq T\}$

(ii) has the same distribution as $(X_t)_{t\geq 0}$ starting from i.

Example

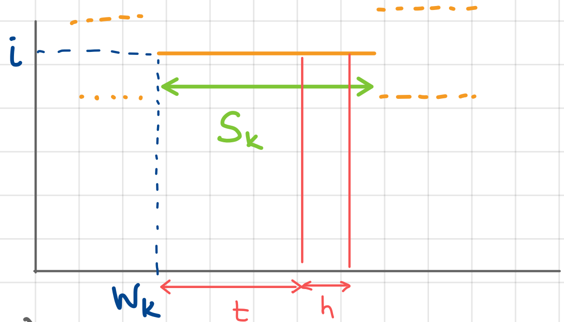
$(X_{W_1+t})_{t\geq 0}$ has the same distribution as $(X_t)_{t\geq 0}$ conditioned on $X_0=i$ and is indep. of what happened before



Alternative (jump and hold) characterization

"Proof"

Denote $G_i(t) := P(S_k > t \mid X_{W_k} = i)$



$$G_i(t+h) = P(S_k > t+h \mid X_{W_k} = i)$$

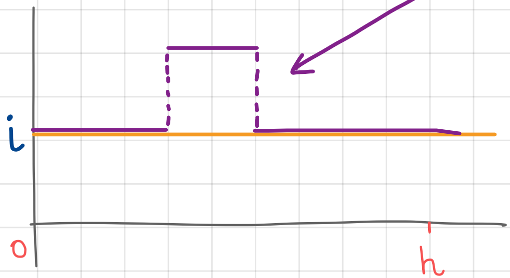
$$\overset{\text{S Markov}}{=} P(\text{no jumps on } [0, t+h] \mid X_0 = i)$$

↑ stopping time

$$\overset{\text{Markov}}{=} P(\text{no jumps on } [0,t] \mid X_0 = i)\, P(\text{no jumps on } [0,h] \mid X_0 = i)$$

$$= P(S_0 > t \mid X_0 = i)\, P(S_0 > h \mid X_0 = i) = G_i(t)\Big(\overbrace{1 - (\lambda_i + \mu_i)h + o(h)}^{P_{ii}(h)}\Big)$$

$$= G_i(t) - (\lambda_i + \mu_i) G_i(t) h + G_i(t) o(h)$$



$\hookrightarrow$

$$G_i'(t) = -(\lambda_i + \mu_i) G_i(t), \quad G_i(0) = 1$$

$P_{i,i+1}(h) = \lambda_i h + o(h) \qquad P_{i+1,i}(h) = \mu_{i+1} h + o(h)$

Alternative (jump and hold) characterization

"Proof" cont.

$$e^x = 1 + x + \frac{x^2}{2} + \cdots \frac{x^n}{n!} + \cdots$$

$$G_i'(t) = -(\lambda_i+\mu_i)\,G_i(t)\,,\quad G_i(0)=1$$

$\hookrightarrow$ $G_i(t) = e^{-(\lambda_i+\mu_i)t} = P(S_k > t \mid X_{W_k} = i)$

✓ $\hookrightarrow$ $S_k \sim \mathrm{Exp}(\lambda_i+\mu_i)$ (given that the process sojourns in i)

Suppose the process waits $\mathrm{Exp}(\lambda_i+\mu_i)$, then jumps to $i+1$ with probability $\lambda_i/(\lambda_i+\mu_i)$

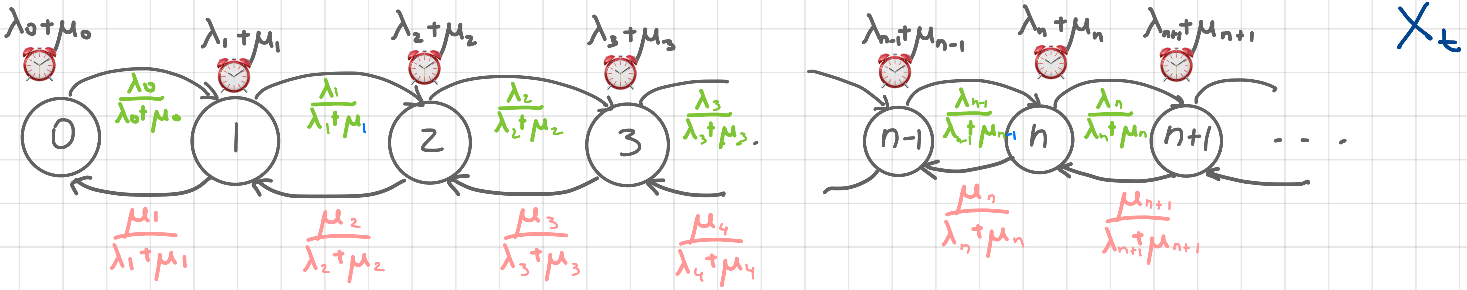
to $i-1$ with probability $\mu_i/(\lambda_i+\mu_i)$

$$P_{i,i+1}(h) \text{"="} P(S_k \le h \mid X_{W_k}=i)\,P(\text{jump to } i+1)$$ ✓

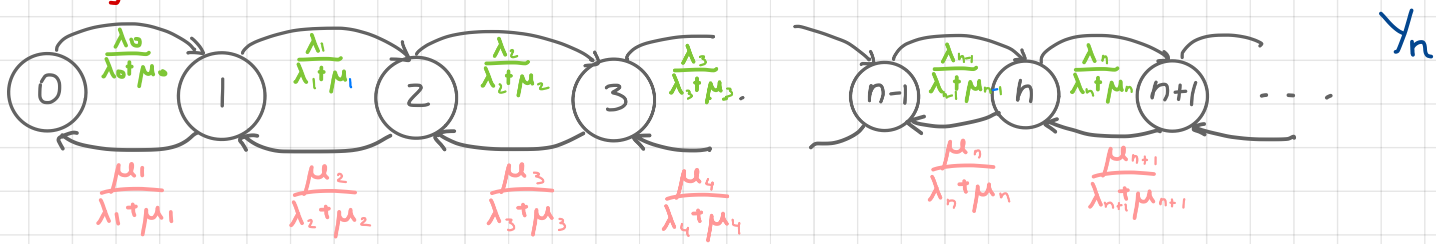
$$= \left(1-e^{-(\lambda_i+\mu_i)h}\right)\frac{\lambda_i}{\lambda_i+\mu_i} = \left((\lambda_i+\mu_i)h + o(h)\right)\frac{\lambda_i}{\lambda_i+\mu_i} = \lambda_i h + o(h)$$

$$P_{i,i-1}(h) \text{"="} P(S_k \le h \mid X_{W_k}=i)\,P(\text{jump to } i-1) = \left((\lambda_i+\mu_i)h + o(h)\right)\frac{\mu_i}{\lambda_i+\mu_i} = \mu_i h + o(h)$$

Related discrete time MC.



Def. Let $(X_t)_{t\geq 0}$ be a continuous time MC, let $W_n, n\geq 0$, be the corresponding waiting (arrival, jump) times. Then we call $(Y_n)_{n\geq 0}$ defined by $Y_n = X_{W_n}$, $Y_0 = X_0$, $n\geq 1$ the jump chain of $(X_t)_{t\geq 0}$.



random walk

Related discrete time MC.

$(X_t)_{t\geq 0}$ and its jump chain $(Y_n)_{n\geq 0}$ execute the same transitions.

Let $(X_t)_{t\geq 0}$ be a birth and death process. Then the transition probability matrix of the random walk $(Y_n)_{n\geq 0}$ is given by

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\ \begin{pmatrix} & \frac{\lambda_0}{\lambda_0+\mu_0} & & & & \cdots \\ \frac{\mu_1}{\lambda_1+\mu_1} & & \frac{\lambda_1}{\lambda_1+\mu_1} & & & \cdots \\ & \frac{\mu_2}{\lambda_2+\mu_2} & & \frac{\lambda_2}{\lambda_2+\mu_2} & & \cdots \\ & & \frac{\mu_3}{\lambda_3+\mu_3} & & \frac{\lambda_3}{\lambda_3+\mu_3} & \cdots \\ & \vdots & & & & \end{pmatrix} \end{array}$$

Absorption probabilities for B&D processes

Let $(X_t)_{t\geq 0}$ be a birth and death process, and assume that the state 0 is absorbing, $\lambda_0 = 0$. Then

$P((X_t)_{t\geq 0}$ gets absorbed in $0 \mid X_0 = i)$

$= P((Y_n)_{n\geq 0}$ gets absorbed in $0 \mid Y_0 = i)$

↳ use the first step analysis to compute the absorption probabilities for $(Y_n)_{n\geq 0}$ (and for $(X_t)_{t\geq 0}$)

Denote $u_i = P(Y_n$ is absorded in $0 \mid Y_0 = i)$

Then
$$u_0 = 1, \quad u_n = \frac{\mu_n}{\lambda_n + \mu_0} u_{n-1} + \frac{\lambda_n}{\lambda_n + \mu_n} u_{n+1}$$

Absorption probabilities for B&D processes

$$u_0 = 1 \quad , \qquad u_n = \frac{\mu_n}{\lambda_n+\mu_n} u_{n-1} + \frac{\lambda_n}{\lambda_n+\mu_n} u_{n+1}$$

Rewrite
$$(\lambda_n+\mu_n) u_n = \mu_n u_{n-1} + \lambda_n u_{n+1}$$

$$\lambda_n (u_{n+1} - u_n) = \mu_n (u_n - u_{n-1})$$

$$u_{n+1} - u_n = \frac{\mu_n}{\lambda_n} (u_n - u_{n-1})$$

$$= \underbrace{\frac{\mu_n}{\lambda_n} \cdot \frac{\mu_{n-1}}{\lambda_{n-1}} \cdot \cdots \cdot \frac{\mu_1}{\lambda_1}}_{\rho_n} (u_1 - \overset{=1}{u_0})$$

$$(*) \quad u_{n+1} - u_n = \rho_n (u_1 - 1)$$

Note that $\sum_{k=1}^{n-1} (u_{k+1} - u_k) = u_n - u_1 = (u_1 - 1) \sum_{n=1}^{n-1} \rho_n$

If $\sum_{n=1}^{\infty} \rho_n = \infty$, then $u_1 = 1$ and from $(*)$ $u_n = 1 \;\; \forall\, n \geq 0$.

Absorption probabilities for B&D processes

Let $\sum_{k=1}^{\infty} \rho_k < \infty$. If we assume that $u_n \to 0$, $n \to \infty$, then by

taking $n \to \infty$

$$\underbrace{u_n}_{0} - u_1 = (u_1 - 1) \sum_{k=1}^{n-1} \rho_k$$

$$u_1 = \frac{\sum_{k=1}^{\infty} \rho_k}{1 + \sum_{k=1}^{\infty} \rho_k}$$

and

$$u_n = u_1 + (u_1 - 1) \sum_{k=1}^{n-1} \rho_k = \frac{\sum_{k=1}^{\infty} \rho_k + \left(\sum_{k=1}^{\infty} \rho_k + 1 - \sum_{k=1}^{\infty} \rho_k \right) \sum_{k=1}^{n-1} \rho_k}{1 + \sum_{k=1}^{\infty} \rho_k}$$

$$= \frac{\sum_{k=1}^{\infty} \rho_k - \sum_{k=1}^{n-1} \rho_k}{1 + \sum_{k=1}^{\infty} \rho_k} = \frac{\sum_{k=n}^{\infty} \rho_k}{1 + \sum_{k=1}^{\infty} \rho_k}$$