MATH180C: Introduction to Stochastic Processes II

Lecture A00: math-old.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math-old.ucsd.edu/~ynemish/teaching/180cB

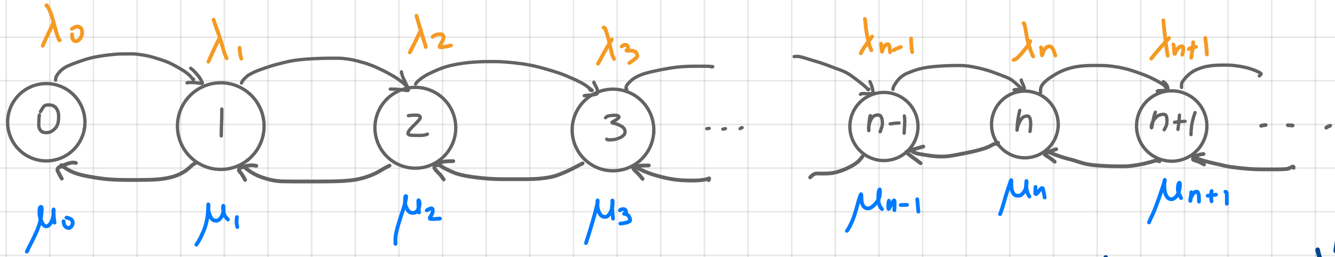
Today: Birth and death processes. Absorption times. General CTMC. Matrix exponentials

Next: PK 6.6, Durrett 4.1

Week 2:

- homework 1 (due Friday April 8)

Alternative (jump and hold) characterization



$\lambda = \mu = 1 \qquad \lambda' = \mu' = 2$

$Z = \frac{\lambda}{\lambda+\mu} = \frac{\mu}{\lambda+\mu} = \frac{\lambda'}{\lambda'+\mu'} = \frac{\mu'}{\lambda'+\mu'}$

$\text{Exp}(2) \qquad \text{Exp}(4)$

Sojourn times S_k are independent,

Each transition has two parts

- wait in state i for time $\sim \text{Exp}(\lambda_i + \mu_i)$
- then choose where to go:

go $\to$ $(i+1)$ with probability $\frac{\lambda_i}{\lambda_i + \mu_i}$

go $(i-1) \leftarrow$ with pobability $\frac{\mu_i}{\lambda_i + \mu_i}$

Stopping times

Def (Informal). Let $(X_t)_{t\geq 0}$ be a stochastic process and let $T \geq 0$ be a random variable. We call T a **stopping time** if the event

$$\{T \leq t\}$$

can be determined from the knowledge of the process up to time t (i.e., from $\{X_s : 0 \leq s \leq t\}$)

Examples: Let $(X_t)_{t\geq 0}$ be right-continuous

1. $\min\{t \geq 0 : X_t = i\}$ is a stopping time
2. W_k is a stopping time
3. $\sup\{t \geq 0 : X_t = i\}$ is not a stopping time

Strong Markov property

Theorem (no proof)

Let $(X_t)_{t\geq 0}$ be a MC, let T be a stopping time of $(X_t)_{t\geq 0}$. Then, conditional on $T<\infty$ and $X_T = i$,

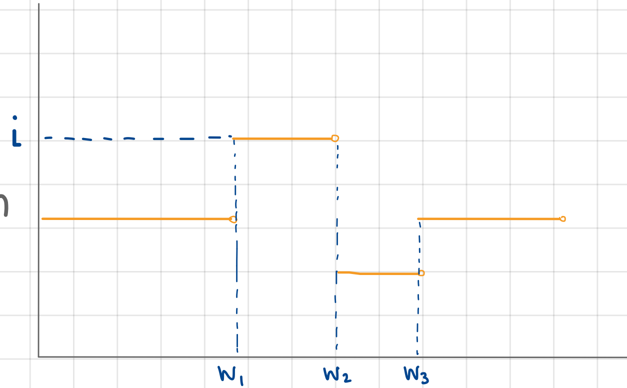
$$(X_{T+t})_{t\geq 0}$$

(i) is independent of $\{X_s,\ 0\leq s\leq T\}$

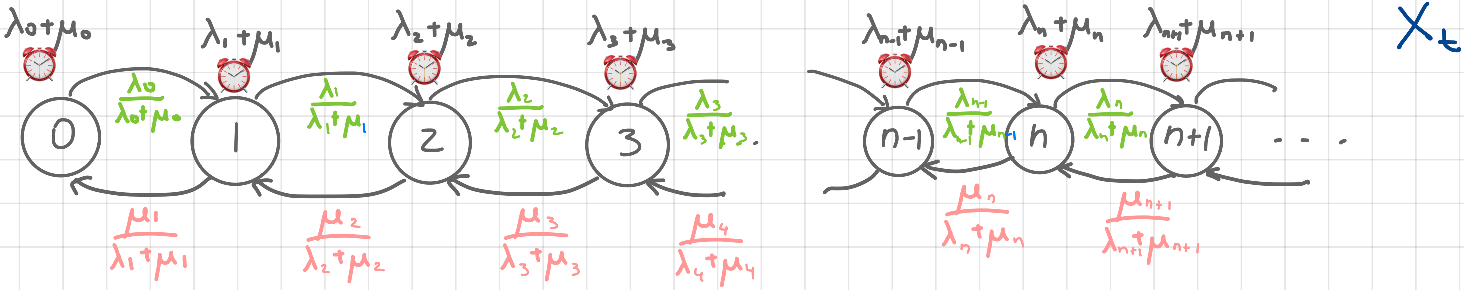
(ii) has the same distribution as $(X_t)_{t\geq 0}$ starting from i.

Example

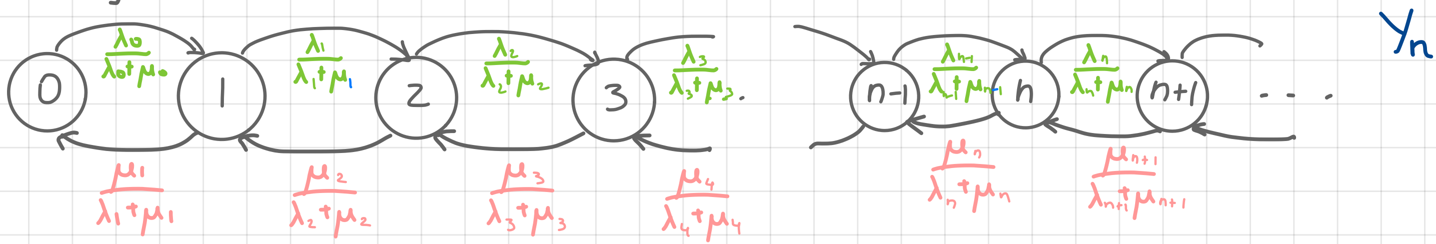
$(X_{W_1+t})_{t\geq 0}$ has the same distribution as $(X_t)_{t\geq 0}$ conditioned on $X_0 = i$ and is indep. of what happened before



Related discrete time MC.



__Def__. Let $(X_t)_{t\geq 0}$ be a continuous time MC, let $W_n, n\geq 0$, be the corresponding waiting (arrival, jump) times. Then we call $(Y_n)_{n\geq 0}$ defined by the jump chain of $(X_t)_{t\geq 0}$.



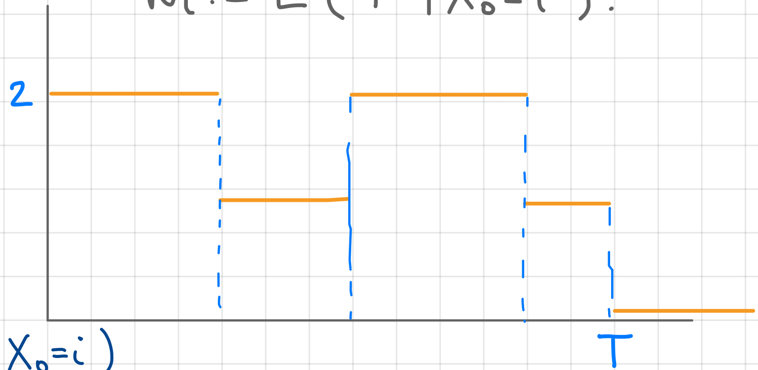
random walk

Mean time until absorption

Let $(X_t)_{t\geq 0}$ be a birth and death process. Denote

$T = \min\{t \geq 0 : X_t = 0\}$ absorption time and $w_i := E(T \mid X_0 = i)$.

Let $(Y_n)_{n\geq 0}$ be the jump chain for $(X_t)_{t\geq 0}$.



$$N := \min\{n \geq 0 : Y_n = 0\}$$

Then $\displaystyle T = \sum_{k=0}^{N-1} S_k$

$$w_i = E\left(\sum_{k=0}^{N-1} S_k \,\middle|\, X_0 = i\right) = \overset{= E(S_0 \mid X_0 = i)}{\frac{1}{\lambda_i + \mu_i}} + E\left(\sum_{k=1}^{N-1} S_k \,\middle|\, X_0 = i\right)$$

$$= \frac{1}{\lambda_i + \mu_i} + \underbrace{E\left(\sum_{k=1}^{N-1} S_k \,\middle|\, X_0 = i, Y_1 = i+1\right)}_{\overset{\text{SMP}}{=} w_{i+1}} P(Y_1 = i+1 \mid Y_0 = i)$$

$$+ \underbrace{E\left(\sum_{k=1}^{N-1} S_k \,\middle|\, X_0 = i, Y_1 = i-1\right)}_{\overset{\text{SMP}}{=} w_{i-1}} P(Y_1 = i-1 \mid Y_0 = i)$$

Mean time until absorption

$$\begin{cases} w_i = \dfrac{1}{\lambda_i+\mu_i} + \dfrac{\lambda_i}{\lambda_i+\mu_i} w_{i+1} + \dfrac{\mu_i}{\lambda_i+\mu_i} w_{i-1}, \\ w_0 = 0 \end{cases}$$

Alternatively, and one can show that

$$E(T \mid X_0 = i) = E\left(\sum_{k=0}^{N-1} \frac{1}{\lambda_{Y_k} + \mu_{Y_k}} \,\Big|\, Y_0 = i \right)$$

Now apply the first step analysis for the general MC

$$w_i = E\Big(\sum_{k=0}^{N-1} g(Y_k) \overset{=1}{} \,\Big|\, Y_0 = i \Big),$$

which leads to (the same) system of equations

$$w_i = g(i) + \sum_{j=1}^{\infty} P_{ij} w_j$$

First step analysis for birth and death processes

Summary:

Let $(X_t)_{t\geq 0}$ be a birth and death process of rates $((\lambda_i, \mu_i))_{i\geq 0}$ with $\lambda_0 = 0$ (state 0 absorbing).

Denote $T = \min\{t : X_t = 0\}$, $u_i = P(X_t \text{ gets absorbed in } 0 \mid X_0 = i)$

$w_i = E(T \mid X_0 = i)$ and $\rho_j = \dfrac{\mu_1 \mu_2 \cdots \mu_j}{\lambda_1 \lambda_2 \cdots \lambda_j}$. Then

$$u_i = \begin{cases} \dfrac{\sum_{j=i}^{\infty} \rho_j}{1 + \sum_{j=1}^{\infty} \rho_j}, & \text{if } \sum_{j=1}^{\infty} \rho_j < \infty \\ 1, & \text{if } \sum_{j=1}^{\infty} \rho_j = \infty \end{cases}$$

$$w_i = \begin{cases} \sum_{j=1}^{\infty} \dfrac{1}{\lambda_j \rho_j} + \sum_{k=1}^{i-1} \rho_k \sum_{j=k+1}^{\infty} \dfrac{1}{\lambda_j \rho_j}, & \text{if } \sum_{j=1}^{\infty} \dfrac{1}{\lambda_j \rho_j} < \infty \\ \infty, & \text{if } \sum_{j=1}^{\infty} \dfrac{1}{\lambda_j \rho_j} = \infty \end{cases}$$

Birth and death processes. Results

- infinitesimal transition probability description
- sojourn time description (jump and hold)

  sojourn times are independent exponential r.v.s

  $$P(i \to i+1) = \frac{\lambda_i}{\lambda_i + \mu_i}, \quad P(i \to i-1) = \frac{\mu_i}{\lambda_i + \mu_i}$$

- system of differential equations for pure birth/death

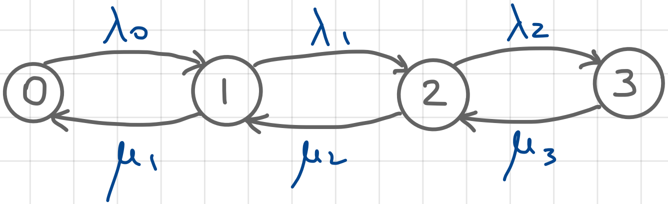
  e.g. $$P_i'(t) = -\lambda_i P_i(t) + \lambda_{i-1} P_{i-1}(t)$$

- distributions of X_t for linear birth (geometric) and linear death (binomial) processes
- first step analysis giving absorption probabilities and mean time to absorption
- explosion, Strong Markov property etc.

General continuous time MC

Assume for simplicity that the state space is finite

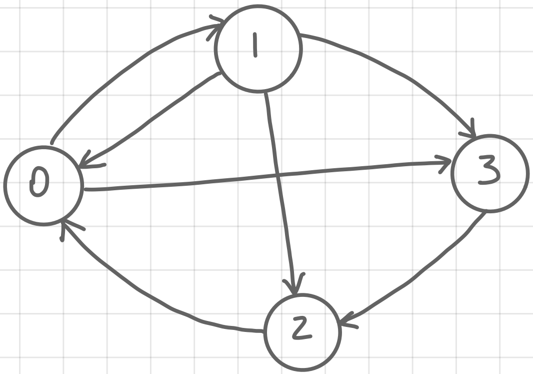
$(X_t)_{t\geq 0}$ is right-continuous



birth and death process

$P_{i,i+1}(h) = \lambda_i h + o(h)$

$P_{i,i}(h) = 1 - \lambda_i h + o(h)$ $\quad h \to 0$



general MC

$P(X_{t+s} = j \mid X_t = i) = P(X_s = j \mid X_0 = i)$

$P_{ij}(h) = q_{ij} h + o(h)$

How to define? How to analyze?

Q-matrices (infinitesimal generators)

Let $S = \{0, 1, \dots, N\}$. We call $Q = (q_{ij})_{i,j=0}^{N}$ a Q-matrix if Q satisfies the following conditions:

(a) $0 \le -q_{ii} < \infty$ for all i

(b) $q_{ij} \ge 0$ for all $i \ne j$

(c) $\sum_j q_{ij} = 0$ for all i

$q_i := \sum_{j \ne i} q_{ij}$

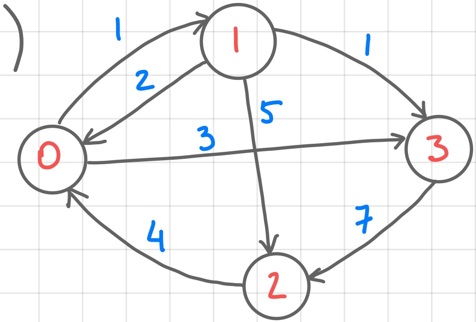
then $q_{ii} = -q_i$

Examples

(a)

$$Q = \begin{pmatrix} -2 & 1 & 1 \\ 2 & -7 & 5 \\ 0 & 2 & -2 \end{pmatrix}$$

(b)



$$\begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} -4 & 1 & 0 & 3 \\ 2 & -8 & 5 & 1 \\ 4 & 0 & -4 & 0 \\ 0 & 0 & 7 & -7 \end{pmatrix} \end{array}$$

Matrix exponentials

Let $Q = (q_{ij})_{i,j=1}^{N}$ be a matrix. Then the series $\sum_{k=0}^{\infty} \frac{Q^k}{k!}$ converges componentwise, and we denote its sum

$$\sum_{k=0}^{\infty} \frac{Q^k}{k!} =: e^Q,$$

the matrix exponential of Q.

In particular, we can define $e^{tQ} = \sum_{k=0}^{\infty} \frac{Q^k t^k}{k!}$ for $t \geq 0$.

<u>Thm.</u> Define $P(t) = e^{tQ}$. Then

(i) for all s, t

(ii) $(P(t))_{t\geq 0}$ is the unique solution to the equations

$$\begin{cases} \frac{d}{dt} P(t) = \\ P(0) = \end{cases}, \quad \text{and} \quad \begin{cases} \frac{d}{dt} P(t) = \\ P(0) = \end{cases}$$