MATH180C: Introduction to Stochastic Processes II

Lecture A00: math-old.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math-old.ucsd.edu/~ynemish/teaching/180cB

Today: General continuous time Markov chains. Matrix exponentials

Next: PK 6.3, 6.6, Durrett 4.2

Week 3:

- homework 2 (due Friday April 15)

Q-matrices (infinitesimal generators)

Let $S = \{0, 1, \dots, N\}$. We call $Q = (q_{ij})_{i,j=0}^{N}$ a Q-matrix if Q satisfies the following conditions:

(a) $0 \le -q_{ii} < \infty$ for all i

(b) $q_{ij} \ge 0$ for all $i \ne j$

(c) $\sum_j q_{ij} = 0$ for all i

$q_i := \sum_{j \ne i} q_{ij}$

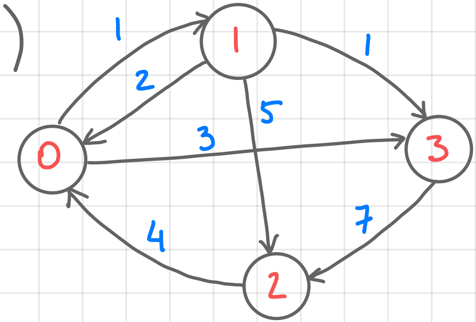
then $q_{ii} = -q_i$

Examples

(a)

$$Q = \begin{pmatrix} -2 & 1 & 1 \\ 2 & -7 & 5 \\ 0 & 2 & -2 \end{pmatrix}$$

(b)



$$\begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} -4 & 1 & 0 & 3 \\ 2 & -8 & 5 & 1 \\ 4 & 0 & -4 & 0 \\ 0 & 0 & 7 & -7 \end{pmatrix} \end{array}$$

Matrix exponentials

Let $Q = (q_{ij})_{i,j=1}^{N}$ be a matrix. Then the series $\sum_{k=0}^{\infty} \frac{Q^k}{k!}$ converges componentwise, and we denote its sum $\sum_{k=0}^{\infty} \frac{Q^k}{k!} =: e^Q$, the matrix exponential of Q.

In particular, we can define $e^{tQ} = \sum_{k=0}^{\infty} \frac{Q^k t^k}{k!}$ for $t \geq 0$.

$$\sum_{k=0}^{\infty} \frac{t^k Q^k}{k!} Q = Q \sum_{k=0}^{\infty} \frac{t^k Q^k}{k!}$$

<u>Thm</u>. Define $P(t) = e^{tQ}$. Then

(i) $P(t+s) = P(t)P(s)$ for all $s, t > 0$

(ii) $(P(t))_{t \geq 0}$ is the unique solution to the equations

$$\begin{cases} \frac{d}{dt} P(t) = P(t) Q \\ P(0) = I \end{cases}, \quad \text{and} \quad \begin{cases} \frac{d}{dt} P(t) = Q P(t) \\ P(0) = I \end{cases}$$

Matrix exponentials

Properties are easy to remember $\longrightarrow$ scalar exponential

(i) $e^{(t+s)Q} = e^{tQ} e^{sQ} = e^{sQ} e^{tQ}$ $\quad \left(e^{(t+s)\alpha} = e^{t\alpha} e^{s\alpha} \right)$

(note that in general $AB \neq BA$ for matrices A, B)

(ii) $\frac{d}{dt} e^{tQ} = Q e^{tQ} = e^{tQ} Q$ $\quad \left(\frac{d}{dt} e^{t\alpha} = \alpha e^{t\alpha} \right)$

$e^{0 \cdot Q} = I$ $\quad (e^0 = 1)$

Example

(a) $Q_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Q_1^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow e^{tQ_1} = I + Q_1 t + \overset{0}{\overset{\shortparallel}{Q_1^2}} \frac{t^2}{2} + \dots = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

(b) $Q_2 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $e^{tQ_2} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$

Matrix exponentials

Results on the previous slide hold for any matrix Q.

<u>Thm</u>. Matrix Q is a Q-matrix

iff $P(t) = e^{tQ}$ is a stochastic matrix $\forall t$

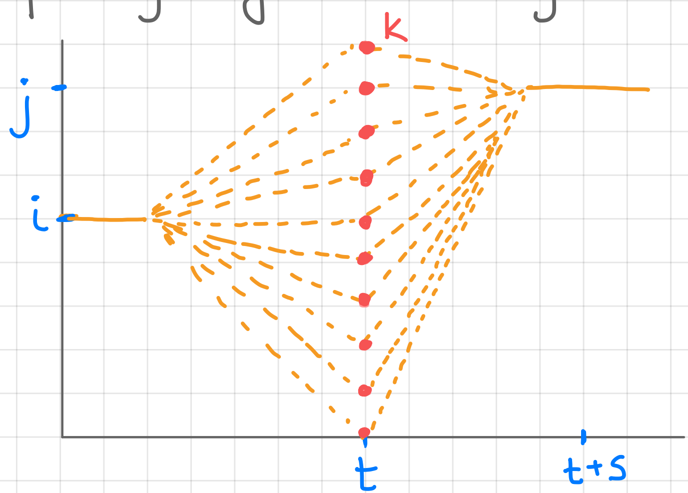
$$\sum_j P_{ij}(t) = 1 \quad \text{for all } i \text{ and } t \geq 0$$

<u>Remarks</u> The semigroup property gives entrywise

$$P_{ij}(t+s) = [P(t)P(s)]_{ij}$$

$$= \sum_{k=0}^{N} P_{ik}(t) P_{kj}(s)$$

(if you think about MC → Chapman-Kolmogorov)



Main theorem

Let $P(t)$ be a matrix-valued function $t \geq 0$.

Consider the following properties

(a) $P_{ij}(t) \geq 0$, $\sum_j P_{ij}(t) = 1$ for all $i, j, t \geq 0$

(b) $P(0) = I$

(c) $P(t+s) = P(t)P(s)$ for all $t, s \geq 0$

(d) $\lim_{t \downarrow 0} P(t) = I$ (continuous at 0)

<u>Theorem A.</u> $P(t)$ satisfies (a)–(d)

if and only if

$P(t) = e^{tQ}$ for some Q-matrix Q

Main theorem. Remarks

This theorem establishes one-to-one correspondance between matrices $P(t)$ satisfying (a)-(d) and the Q-matrices of the same dimension.

Remarks

1. Conditions (a)-(d) imply that $P(t)$ is differentiable

2. If $P(t) = e^{tQ}$, then $P(h) = I + Qh + o(h)$ as $h \to 0$

$$P(h) = I + Qh + \underbrace{\sum_{k \geq 2} \frac{Q^k h^k}{k!}}_{= o(h)}$$

Q-matrices and Markov chains

Let $(X_t)_{t\geq 0}$ be a continuous time MC, $X_t \in \{0,1,\dots,N\}$ with right-continuous sample paths

Denote $P_{ij}(t) = P(X_t = j \mid X_0 = i)$, $i,j \in \{0,1,\dots,N\}$

stationary

Then

- $P_{ij}(t) \geq 0$, $\sum_{j=0}^{N} P_{ij}(t) = 1 \quad \left(= \sum_{j=0}^{N} P(X_t = j \mid X_0 = i) \right)$

- $P_{ij}(0) = \delta_{ij} \quad \left(P(X_0 = j \mid X_0 = i) = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases} \right) \iff P(0) = I$

- $P_{ij}(t+s) = P(X_{t+s} = j \mid X_0 = i) \quad = \sum_{k=0}^{N} P_{kj}(s) P_{ik}(t)$

$$= \sum_{k=0}^{N} P(X_{t+s} = j \mid X_0 = i, X_t = k) P(X_t = k \mid X_0 = i)$$

- $\lim_{h \downarrow 0} P(X_h = j \mid X_0 = i) = \delta_{ij}$

$P(h) \to I, \ h \to 0$

Q-matrices and Markov chains (cont.)

$P(t)$ satisfies properties (a)-(d) from Theorem A.

$\Longrightarrow$ there is a Q-matrix Q such that

$$P(t) = e^{tQ}$$

$$p_{ij}(h) = q_{ij}h + o(h) \quad i \neq j$$

$$p_{ii}(h) = 1 + q_{ii}h + o(h)$$

In particular,

$$P(h) = I + Qh + o(h) \quad \text{as } h \to 0$$

This implies the one-to-one correspondance between Q-matrices and continuous time MC with right-continuous sample paths.

Q is called the infinitesimal generator of $(X_t)_{t \geq 0}$

Infinitesimal description of cont. time MC

Let $Q = (q_{ij})_{i,j=0}^{N}$ be a Q-matrix, let $(X_t)_{t\geq 0}$ be right-continuous stochastic process, $X_t \in \{0, 1, \dots, N\}$.

We call $(X_t)_{t\geq 0}$ a Markov chain with generator Q, if

(i) $(X_t)_{t\geq 0}$ satisfies the Markov property

(ii) $$P(X_{t+h} = j \mid X_t = i) = \begin{cases} q_{ij}h + o(h), & i \neq j \\ 1 + q_{ii}h + o(h), & i = j \end{cases} \quad \text{as } h \to 0$$

Example

Pure death process

- $P_{i,i-1}(h) = \mu_i h + o(h)$
- $P_{ii}(h) = 1 - \mu_i h + o(h)$
- $P_{ij}(h) = o(h)$ for $j \notin \{i-1, i\}$

The corresponding Q-matrix

$$Q = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ N \end{array} \begin{pmatrix} 0 & 0 & 0 & \cdots & & \\ \mu_1 & -\mu_1 & 0 & \cdots & & \\ 0 & \mu_2 & -\mu_2 & 0 & \cdots & \\ & & \ddots & \ddots & & \\ & & & & \ddots & \\ & & & & \mu_N & -\mu_N \end{pmatrix}$$

Sojourn time description

Let $Q = (q_{ij})_{i,j=0}^{N}$ be a Q-matrix. Denote $q_i = \sum_{j \neq i} q_{ij}$

so that

$$Q = \begin{pmatrix} -q_0 & q_{01} & q_{02} & \cdots \\ q_{10} & -q_1 & q_{12} & \cdots \\ q_{20} & q_{21} & -q_2 & \cdots \\ \vdots & \vdots & & \end{pmatrix} \qquad q_0 = \sum_{i \neq 0} q_{0i} \quad \vdots$$

Denote $Y_k := X_{W_k}$ (jump chain).

Then the MC with generator matrix Q has the following equivalent jump and hold description

- sojourn times S_k are independent r.v. with $P(S_k > t \mid Y_k = i) = e^{-q_i t}$ $\left(S_k \sim \mathrm{Exp}(q_i) \right)$
- transition probabilities $P(Y_{k+1} = j \mid Y_k = i) = \dfrac{q_{ij}}{q_i}$