MATH180C: Introduction to Stochastic Processes II

Lecture A00: math-old.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math-old.ucsd.edu/~ynemish/teaching/180cB

Today: General continuous time Markov chains. Matrix exponentials

Next: PK 6.3, 6.6, Durrett 4.2

Week 3:

- homework 2 (due Friday April 15)

Q-matrices (infinitesimal generators)

Let $S = \{0, 1, \dots, N\}$. We call $Q = (q_{ij})_{i,j=0}^{N}$ a Q-matrix if Q satisfies the following conditions:

(a) $0 \le -q_{ii} < \infty$ for all i

(b) $q_{ij} \ge 0$ for all $i \ne j$

(c) $\sum_j q_{ij} = 0$ for all i

$q_i := \sum_{j \ne i} q_{ij}$

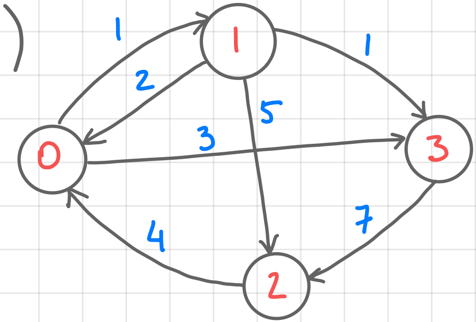
then $q_{ii} = -q_i$

Examples

(a)

$$Q = \begin{pmatrix} -2 & 1 & 1 \\ 2 & -7 & 5 \\ 0 & 2 & -2 \end{pmatrix}$$

(b)



$$\begin{array}{c|c} & \begin{matrix} 0 & \;\;1 & \;\;2 & \;\;3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} -4 & 1 & 0 & 3 \\ 2 & -8 & 5 & 1 \\ 4 & 0 & -4 & 0 \\ 0 & 0 & 7 & -7 \end{pmatrix} \end{array}$$

Matrix exponentials

Let $Q = (q_{ij})_{i,j=1}^{N}$ be a matrix. Then the series $\sum_{k=0}^{\infty} \frac{Q^k}{k!}$ converges componentwise, and we denote

its sum $\sum_{k=0}^{\infty} \frac{Q^k}{k!} =: e^Q$, the matrix exponential of Q.

In particular, we can define $e^{tQ} = \sum_{k=0}^{\infty} \frac{Q^k t^k}{k!}$ for $t \geq 0$.

Thm. Define $P(t) = e^{tQ}$. Then

(i) for all s, t

(ii) $(P(t))_{t\geq 0}$ is the unique solution to the equations

$$\begin{cases} \frac{d}{dt} P(t) = \\ P(0) = . \end{cases}, \quad \text{and} \quad \begin{cases} \frac{d}{dt} P(t) = \\ P(0) = \end{cases}$$

Matrix exponentials

Properties are easy to remember $\longrightarrow$ scalar exponential

(i) $e^{(t+s)Q} = e^{tQ} e^{sQ} = e^{sQ} e^{tQ}$ $\quad \left(e^{(t+s)\alpha} = e^{t\alpha} e^{s\alpha} \right)$

(note that in general $AB \neq BA$ for matrices A, B)

(ii) $\frac{d}{dt} e^{tQ} = Q e^{tQ} = e^{tQ} Q$ $\quad \left(\frac{d}{dt} e^{t\alpha} = \alpha e^{t\alpha} \right)$

$e^{0 \cdot Q} = I \quad (e^0 = 1)$

Example

(a) $Q_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,

(b) $Q_2 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

Matrix exponentials

Results on the previous slide hold for any matrix Q.

<u>Thm</u>. Matrix Q is a Q-matrix

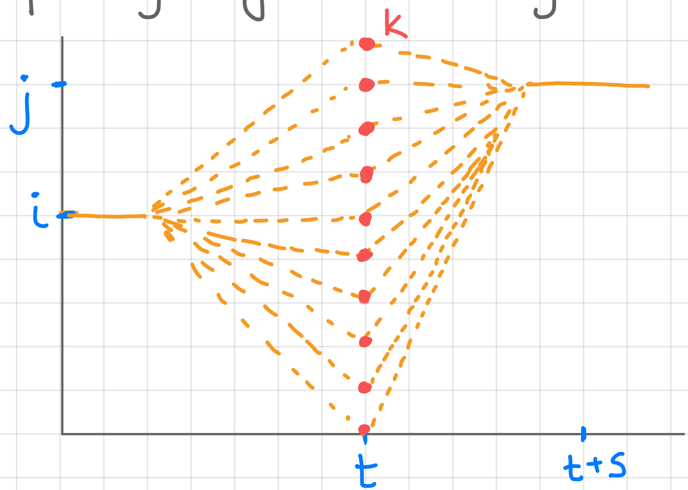
iff $P(t) = e^{tQ}$ is a stochastic matrix $\forall t$

<u>Remarks</u> The semigroup property gives entrywise

$$P_{ij}(t+s) = [P(t)P(s)]_{ij}$$

(if you think about MC $\longrightarrow$

Chapman-Kolmogorov)



Main theorem

Let $P(t)$ be a matrix-valued function $t \geq 0$.

Consider the following properties

(a) $P_{ij}(t) \geq 0$, $\sum_j P_{ij}(t) = 1$ for all $i, j, t \geq 0$

(b) $P(0) = I$

(c) $P(t+s) = P(t)P(s)$ for all $t, s \geq 0$

(d) $\lim_{t \downarrow 0} P(t) = I$ (continuous at 0)

Theorem A. $P(t)$ satisfies (a)–(d)

if and only if

Main theorem. Remarks

This theorem establishes one-to-one correspondance between matrices $P(t)$ satisfying (a) – (d) and the Q-matrices of the same dimension.

Remarks

1. Conditions (a)-(d) imply that $P(t)$ is differentiable

2. If $P(t) = e^{tQ}$, then $P(h) =$ as $h \to 0$

$$P(h) =$$

Q-matrices and Markov chains

Let $(X_t)_{t\geq 0}$ be a continuous time MC, $X_t \in \{0,1,\dots,N\}$
with right-continuous sample paths

Denote $P_{ij}(t) = P(X_t = j \mid X_0 = i)$, $i,j \in \{0,1,\dots,N\}$

stationary

Then

- $P_{ij}(t)$, $\sum_{j=0}^{N} P_{ij}(t)$ $\left(= \sum_{j=0}^{N} P(X_t = j \mid X_0 = i) \right)$
- $P_{ij}(0) =$ $\left(P(X_0 = j \mid X_0 = i) = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases} \right)$
- $P_{ij}(t+s) = P(X_{t+s} = j \mid X_0 = i)$
- $\lim_{h \downarrow 0} P(X_h = j \mid X_0 = i) =$

Q-matrices and Markov chains (cont.)

$P(t)$ satisfies properties (a)-(d) from Theorem A.

$\Longrightarrow$ there is a Q-matrix Q such that

$$P(t) =$$

In particular,

$$P(h) =$$

This implies the one-to-one correspondance between Q-matrices and continuous time MC with right-continuous sample paths.

Q is called the infinitesimal generator of $(X_t)_{t\geq 0}$

Infinitesimal description of cont. time MC

Let $Q = (q_{ij})_{i,j=0}^{N}$ be a Q-matrix, let $(X_t)_{t\geq 0}$ be right-continuous stochastic process, $X_t \in \{0, 1, \dots, N\}$.

We call $(X_t)_{t\geq 0}$ a Markov chain with generator Q, if

(i) $(X_t)_{t\geq 0}$ satisfies the Markov property

(ii) $P(X_{t+h} = j \mid X_t = i) =$

Example

Pure death process

- $P_{i,i-1}(h) = \mu_i h + o(h)$
- $P_{ii}(h) = 1 - \mu_i h + o(h)$
- $P_{ij}(h) = o(h)$ for $j \notin \{i-1, i\}$

The corresponding Q-matrix

$$Q = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

Sojourn time description

Let $Q = (q_{ij})_{i,j=0}^{N}$ be a Q-matrix. Denote $q_i = \sum_{j \neq i} q_{ij}$

so that

$$Q = \begin{pmatrix} & q_{01} & q_{02} & \cdots \\ q_{10} & & q_{12} & \cdots \\ q_{20} & q_{21} & & \cdots \\ \vdots & \vdots & & \ddots \end{pmatrix} \qquad \begin{matrix} q_0 = \sum_{i \neq 0} q_{0i} \\ \vdots \end{matrix}$$

Denote $Y_k := X_{W_k}$ (jump chain).

Then the MC with generator matrix Q has the following equivalent jump and hold description

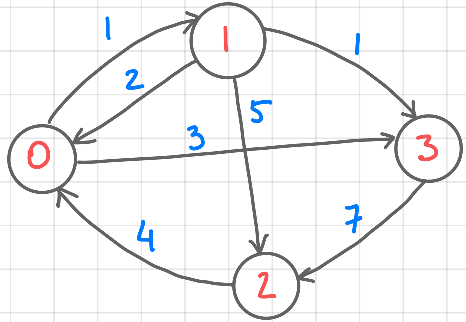
- sojourn times S_k are independent r.v.

  with $P(S_k > t \mid Y_k = i) =$

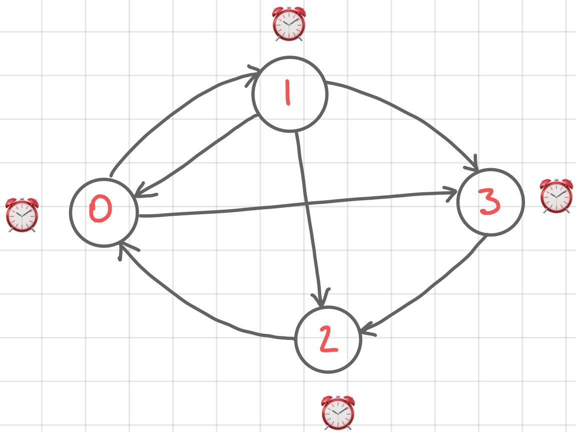
- transition probabilities $P(Y_{k+1} = j \mid Y_k = i) =$

Example

$$
\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c}
\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{pmatrix}
-4 & 1 & 0 & 3 \\
2 & -7 & 5 & 1 \\
4 & 0 & -4 & 0 \\
0 & 0 & 7 & -7
\end{pmatrix}
\end{array}
$$

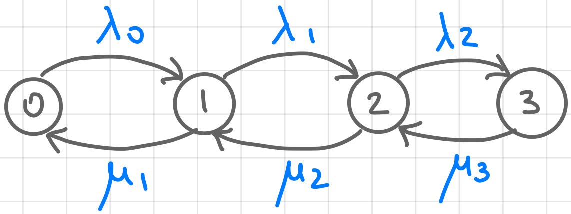


$i \xrightarrow{\alpha} j$ "="



Example

Birth and death process on $\{0, 1, 2, 3\}$



$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & \\ & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 \\ & & \mu_3 & -\mu_3 \end{pmatrix}$$