MATH180C: Introduction to Stochastic Processes II

Lecture A00: math-old.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math-old.ucsd.edu/~ynemish/teaching/180cB

Today: Kolmogorov's equations

Next: PK 6.4, 6.6, Durrett 4.3

Week 3:

- homework 2 (due Friday April 15)
- Midterm 1 date changed: Friday, April 22

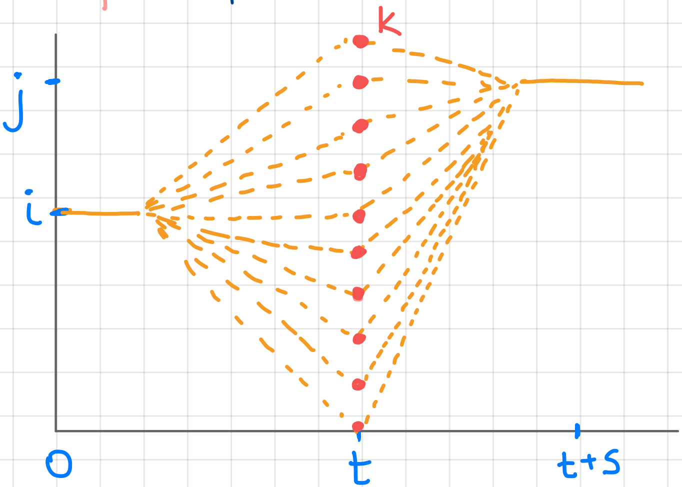
Chapman-Kolmogorov equation

$$P_{ij}(t+s) = P(X_{t+s}=j \mid X_0=i)$$ condition on the value of X_t

$$= \sum_{k=0}^{N} P(X_{t+s}=j \mid X_0=i, X_t=k)\, P(X_t=k \mid X_0=i)$$

Markov $$= \sum_{k=0}^{N} P(X_{t+s}=j \mid X_t=k)\, P(X_t=k \mid X_0=i)$$

stationary trans. prob. $$= \sum_{k=0}^{N} P(X_s=j \mid X_0=k)\, P(X_t=k \mid X_0=i) = \sum_{k=0}^{N} P_{ik}(t)\, P_{kj}(s)$$



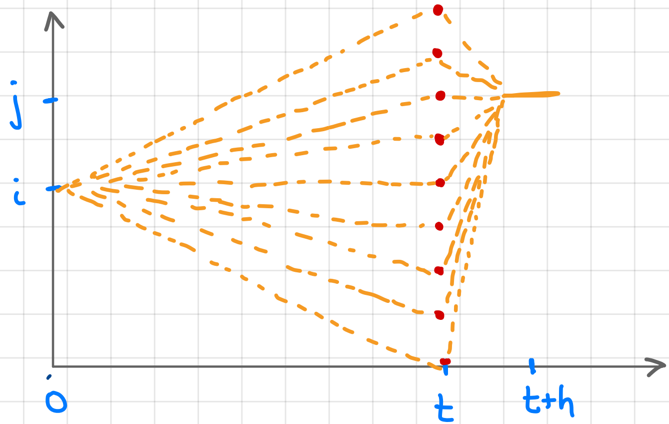
Or in matrix form

$$P(t+s) = P(t)\,P(s)$$

Kolmogorov forward equations

Apply Chapman-Kolmogorov equations to compute

$P_{ij}(t+h)$:



$$P_{ij}(t+h) = \sum_{k=0}^{N} P_{ik}(t)\, P_{kj}(h) \quad (*)$$

Use infinitesimal description:

$P(h) = I + Qh + o(h)$

$$P_{kj}(h) = \begin{cases} q_{kj}\, h + o(h), & k \neq j \\ 1 + q_{jj}\, h + o(h), & k = j \end{cases}$$

$$(*) = P_{ij}(t)\big(1 + q_{jj}\, h + o(h)\big) + \sum_{\substack{k=0 \\ k\neq j}}^{N} P_{ik}(t)\big(q_{kj}\, h + o(h)\big) + o(h)$$

$$= P_{ij}(t) + \underbrace{\sum_{k=0}^{N} P_{ik}(t)\, q_{kj}}_{[P(t)Q]_{ij}} h + o(h)$$

$\Rightarrow \quad P(t+h) = P(t) + P(t)Qh$

$$\boxed{\frac{d}{dt}P(t) = P(t)Q}$$

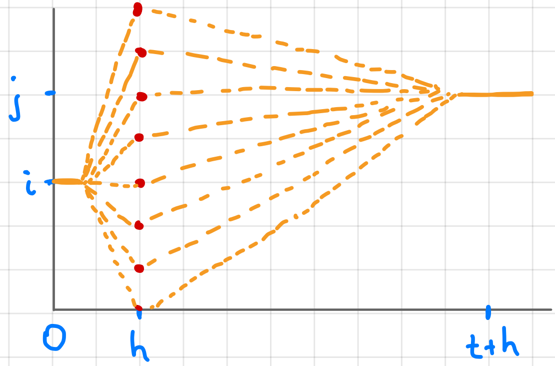
Kolmogorov backward equations

$$P_{ij}(t+h) = \sum_{k=0}^{N} P_{ik}(h)\, P_{kj}(t)$$

$$= \big(1 + q_{ii} h + o(h)\big) P_{ij}(t)$$

$$+ \sum_{\substack{k=0 \\ k\neq i}}^{N} \big(q_{ik} h + o(h)\big) P_{kj}$$

$$= P_{ij}(t) + \overbrace{\sum_{k=0}^{N} q_{ik} P_{kj}(t)}^{[Q P(t)]_{ij}} h + o(h)$$



$$\hookrightarrow \quad \boxed{\frac{d}{dt} P(t) = Q\, P(t)} \qquad P(0) = I$$

Kolmogorov equations. Remarks

1. e^{tQ} satisfies both (forward and backward) equations.

Indeed, omitting technical details, differentiate term-by-term

$$\frac{d}{dt} e^{tQ} = \frac{d}{dt}\left(\sum_{k=0}^{\infty} \frac{Q^k t^k}{k!}\right) = \sum_{k=1}^{\infty} \frac{Q^k t^{k-1}}{(k-1)!} = \sum_{\ell=0}^{\infty} \frac{Q^{\ell+1} t^{\ell}}{\ell!} = Q \sum_{\ell=0}^{\infty} \frac{Q^\ell t^\ell}{\ell!}.$$

$$= \sum_{\ell=0}^{\infty} \frac{Q^\ell t^\ell}{\ell!} Q$$

Now $\displaystyle\sum_{k=1}^{\infty} \frac{Q^k}{(k-1)!} t^{k-1} \overset{\ell=k-1}{=} \sum_{\ell=0}^{\infty} \frac{Q^{\ell+1}}{\ell!} t^\ell = Q e^{tQ} = e^{tQ} Q$

2. Redundancy is related to the stationarity of transition probabilities. If transition probabilities $P_{ij}(s,t) = P(X_t = j \mid X_s = i)$ are not stationary, then

$\frac{\partial}{\partial t} P_{ij}(s,t) \to$ forward equation, $\frac{\partial}{\partial s} P_{ij}(s,t) \to$ backward equation

Example

Two-state MC

$$Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$$

$$Q^2 = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}\begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix} = \begin{pmatrix} \alpha(\alpha+\beta) & -\alpha(\alpha+\beta) \\ -\beta(\alpha+\beta) & \beta(\alpha+\beta) \end{pmatrix} = -(\alpha+\beta) Q$$

$$\hookrightarrow \quad Q^k = (-1)^{k-1}(\alpha+\beta)^{k-1} Q \quad , \quad k \geq 1$$

$$e^{tQ} = \sum_{k=0}^{\infty} \frac{Q^k t^k}{k!} = I + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(\alpha+\beta)^{k-1} t^k}{k!} Q$$

$$= I - \frac{1}{\alpha+\beta} \sum_{k=1}^{\infty} \frac{(-(\alpha+\beta))^k t^k}{k!} Q$$

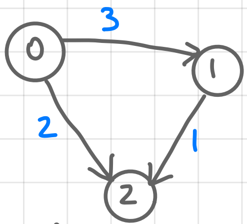
$$= I - \frac{1}{\alpha+\beta}\left(e^{-(\alpha+\beta)t} - 1 \right) Q$$

$$= I + \frac{1}{\alpha+\beta} Q - \frac{1}{\alpha+\beta} e^{-(\alpha+\beta)t} Q$$

Example

Let $(X_t)_{t\geq 0}$ be a MC with generator Q

$$Q = \begin{pmatrix} -5 & 3 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$



Compute $P_{01}(t)$

For any k, $Q^k = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$, $\Rightarrow P_{10}(t) = P_{20}(t) = P_{21}(t) = 0$

$$P'(t) = \begin{pmatrix} P_{00} & P_{01} & P_{02} \\ 0 & P_{11} & P_{12} \\ 0 & 0 & P_{22} \end{pmatrix} \begin{pmatrix} -5 & 3 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$P_{00}'(t) = -5P_{00}(t)$, $P_{00}(0) = 1 \Rightarrow P_{00}(t) = e^{-5t}$

$P_{11}'(t) = -P_{11}(t)$, $P_{11}(0) = 1 \Rightarrow P_{11}(t) = e^{-t}$

$P_{22}'(t) = 0$, $P_{22}(0) = 1 \Rightarrow P_{22}(t) = 1$

$P_{01}'(t) = 3P_{00}(t) - P_{01}(t)$

$P_{01}(0) = 0$

$P_{01}'(t) =$ ⋮

$P_{01}(t) =$ ⋮

$$P_{01}(t) = \frac{3}{4}\left(e^{-t} - e^{-5t}\right)$$

Forward and backward equations for B&D processes

Forward equation:

$$P_{ij}(t+h) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(h)$$

$$= P_{ij}(t)\big(1-(\lambda_j+\mu_j)h+o(h)\big)$$
$$+ P_{i,j-1}(t)\big(\lambda_{j-1}h+o(h)\big) + P_{i,j+1}(t)\big(\mu_{j+1}h+o(h)\big)$$
$$+ \underbrace{\sum_{\substack{k=0 \\ |k-j|>1}}^{\infty} P_{ik}(t)\, o_{kj}(h)}_{\theta_{ij}(h)}$$

If $\theta_{ij} = o(h)$ (requires additional technical assumptions)

$$\begin{cases} P_{ij}'(t) = \lambda_{j-1} P_{i,j-1}(t) - (\lambda_j+\mu_j) P_{ij}(t) + \mu_{j+1} P_{i,j+1}(t) \\ P_{i0}'(t) = -\lambda_0 P_{i0}(t) + \mu_1 P_{i1}(t) \end{cases} \quad , \quad \text{with } P_{ij}(0) = \delta_{ij}$$

Forward and backward equations for B&D processes

Similarly, we derive the backward equations

$$\begin{cases} P_{ij}'(t) = \mu_i P_{i-1,j}(t) - (\lambda_i + \mu_i) P_{ij}(t) + \lambda_i P_{i+1,j}(t) \\ P_{0j}'(t) = -\lambda_0 P_{0j}(t) + \lambda_0 P_{1j}(t) \end{cases}, \quad \text{with } P_{ij}(0) = \delta_{ij}$$

Example Linear growth with immigration.

Recall

$$\lambda_k = \underbrace{\lambda \cdot k}_{\text{linear birth rate}} + \underbrace{a}_{\text{immigration}}$$

$$\mu_k = \underbrace{\mu \cdot k}_{\text{linear death rate}}$$

Example: Linear growth with immigration.

Use forward equations to compute $E(X_t \mid X_0 = i)$

$$\begin{cases} P_{ij}'(t) = \lambda_{j-1} P_{i,j-1}(t) - (\lambda_j + \mu_j) P_{ij}(t) + \mu_{j+1} P_{i,j+1}(t) \\ P_{i0}'(t) = -\lambda_0 P_{i0}(t) + \mu_1 P_{i1}(t) \end{cases}$$

$$M'(t) = \sum_{j=0}^{\infty} j \cdot P_{ij}'(t)$$

$$E(X_t \mid X_0 = i) = \sum_{j=0}^{\infty} j \cdot P(X_t = j \mid X_0 = i) = \sum_{j=0}^{\infty} j \cdot P_{ij}(t) =: M(t)$$

$$P_{ij}'(t) = (\lambda(j-1) + a) P_{i,j-1}(t) - ((\lambda+\mu) j + a) P_{ij}(t) + \mu(j+1) P_{i,j+1}(t)$$

Example: Linear growth with immigration.

$$M'(t) = \cdots \qquad \begin{pmatrix} \quad \end{pmatrix} \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & \cdots & 0 \\ \mu_1 & -(\mu_1+\lambda_1) & \lambda_1 & & \end{pmatrix} \begin{pmatrix} P_{00} & P_{01} & \cdots \\ P_{10} & P_{11} & \end{pmatrix}$$

$$= --$$

$$P_{0j}' = -\lambda_0 P_{0j} + \lambda_0 P_{1j}$$

$$= (\lambda-\mu) M(t) + a$$

$$\begin{cases} M'(t) = (\lambda-\mu) M(t) + a \\ M(0) = i \end{cases}$$

$$M(t) = i + at \quad \text{if } \lambda = \mu$$

$$M(t) = \frac{a}{\lambda-\mu}\left(e^{(\lambda-\mu)t} - 1\right) + i e^{(\lambda-\mu)t} \quad \text{if } \lambda \neq \mu$$