MATH180C: Introduction to Stochastic Processes II

Lecture A00: math-old.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math-old.ucsd.edu/~ynemish/teaching/180cB

Today: Kolmogorov's equations

Next: PK 6.4, 6.6, Durrett 4.3

Week 3:

- homework 2 (due Friday April 15)
- Midterm 1 date changed: Friday, April 22

Chapman-Kolmogorov equation

$$P_{ij}(t+s) = P(X_{t+s}=j \mid X_0=i)$$

condition on the value of X_t

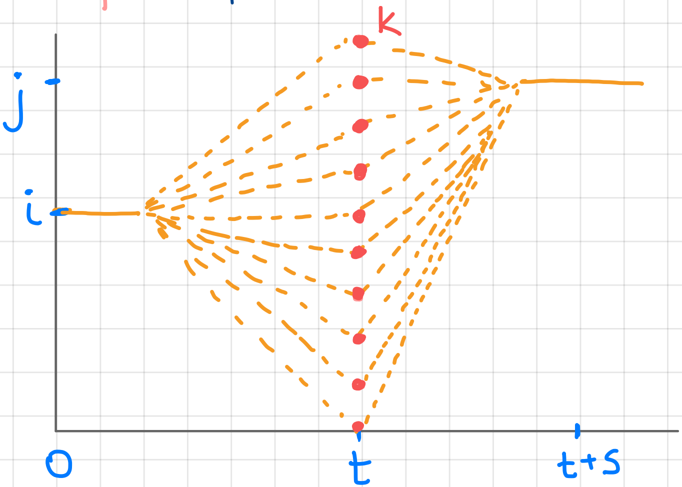
$$= \sum_{k=0}^{N} P(X_{t+s}=j \mid X_0=i, X_t=k)\, P(X_t=k \mid X_0=i)$$

Markov

$$= \sum_{k=0}^{N} P(X_{t+s}=j \mid X_t=k)\, P(X_t=k \mid X_0=i)$$

stationary trans. prob.

$$= \sum_{k=0}^{N} P(X_s=j \mid X_0=k)\, P(X_t=k \mid X_0=i) = \sum_{k=0}^{N} P_{ik}(t)\, P_{kj}(s)$$



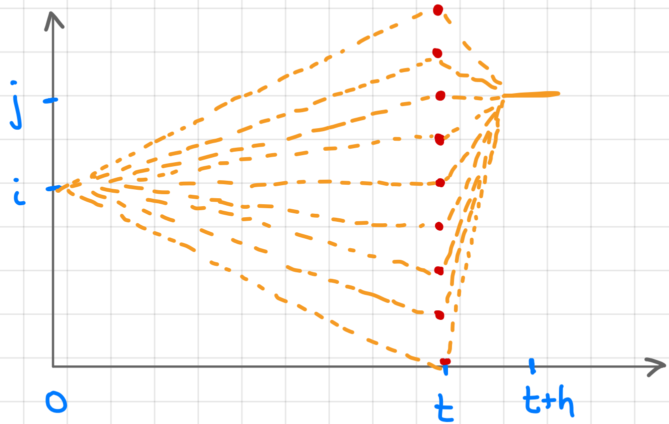
Or in matrix form

$$P(t+s) = P(t)\,P(s)$$

Kolmogorov forward equations

Apply Chapman-Kolmogorov equations to compute

$P_{ij}(t+h)$:



$P_{ij}(t+h) =$

Use infinitesimal description:

$$P_{kj}(h) = \begin{cases} q_{kj}\, h + o(h), & k \neq j \\ 1 + q_{jj}\, h + o(h), & k = j \end{cases}$$

$(*) =$

$=$

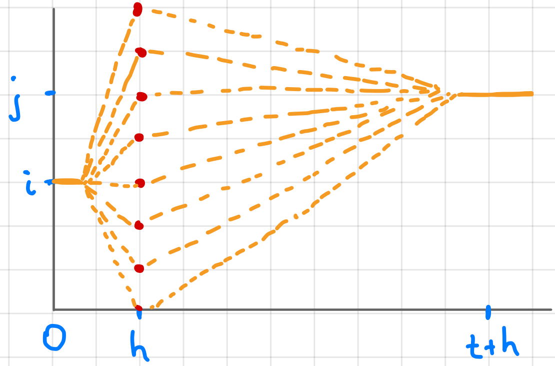
$$\boxed{\frac{d}{dt} P(t) = P(t) Q}$$

Kolmogorov backward equations

$$P_{ij}(t+h) = \sum_{k=0}^{N} P_{ik}(h) P_{kj}(t)$$

$$= \left(1 + q_{ii} h + o(h)\right) P_{ij}(t)$$

$$+ \sum_{\substack{k=0 \\ k \neq i}}^{N} \left(q_{ik} h + o(h)\right) P_{kj}$$



$$= P_{ij}(t) + \sum_{k=0}^{N} q_{ik} P_{kj}(t) h + o(h)$$

↳ $\boxed{}$

Kolmogorov equations. Remarks

1. e^{tQ} satisfies both (forward and backward) equations.

Indeed, omitting technical details, differentiate term-by-term

$$\frac{d}{dt} e^{tQ} = \frac{d}{dt}\left(\sum_{k=0}^{\infty} \frac{Q^k t^k}{k!} \right) =$$

Now $\displaystyle\sum_{k=1}^{\infty} \frac{Q^k}{(k-1)!} t^{k-1} \overset{\ell=k-1}{=} \sum_{\ell=0}^{\infty} \frac{Q^{\ell+1}}{\ell!} t^{\ell} =$

2. Redundancy is related to the stationarity of transition probabilities. If transition probabilities $P_{ij}(s,t) = P(X_t = j \mid X_s = i)$ are not stationary, then

$\frac{\partial}{\partial t} P_{ij}(s,t) \to$ forward equation, $\frac{\partial}{\partial s} P_{ij}(s,t) \to$ backward equation

Example

Two-state MC

$$Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$$

$$Q^2 = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}\begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix} = \begin{pmatrix} \alpha(\alpha+\beta) & -\alpha(\alpha+\beta) \\ -\beta(\alpha+\beta) & \beta(\alpha+\beta) \end{pmatrix} =$$

$\hookrightarrow$

$$e^{tQ} = \sum_{k=0}^{\infty} \frac{Q^k t^k}{k!} =$$

$$=$$

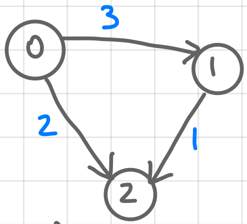
$$=$$

$$= I + \frac{1}{\alpha+\beta} Q - \frac{1}{\alpha+\beta} e^{-(\alpha+\beta)t} Q$$

Example

Let $(X_t)_{t\geq 0}$ be a MC with generator Q

$$Q = \begin{pmatrix} -5 & 3 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$



Compute $P_{01}(t)$

For any k, $Q^k = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$, $\Rightarrow$

$$P'(t) = \begin{pmatrix} P_{00} & P_{01} & P_{02} \\ 0 & P_{11} & P_{12} \\ 0 & 0 & P_{22} \end{pmatrix} \begin{pmatrix} -5 & 3 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$P_{00}'(t) =$

$P_{11}'(t) = -P_{11}(t)$, $P_{11}(0) = 1 \Rightarrow P_{11}(t) = e^{-t}$

$P_{22}'(t) = 0$, $P_{22}(0) = 1 \Rightarrow P_{22}(t) = 1$

$P_{01}'(t) =$

$P_{01}(t) =$

$P_{01}(t) =$

Forward and backward equations for B&D processes

Forward equation:

$$P_{ij}(t+h) = \sum_{k=0}^{\infty} P_{ik}(t)\, P_{kj}(h)$$

$$=$$

If $\theta_{ij} = o(h)$ (requires additional technical assumptions)

$$\begin{cases} P_{ij}'(t) = \lambda_{j-1} P_{i,j-1}(t) - (\lambda_j + \mu_j) P_{ij}(t) + \mu_{j+1} P_{i,j+1}(t) \\ P_{i0}'(t) = -\lambda_0 P_{i0}(t) + \mu_1 P_{i1}(t) \end{cases}, \quad \text{with } P_{ij}(0) = \delta_{ij}$$

Forward and backward equations for B&D processes

Similarly, we derive the backward equations

$$\begin{cases} P'_{ij}(t) = \mu_i P_{i-1,j}(t) - (\lambda_i + \mu_i) P_{ij}(t) + \lambda_i P_{i+1,j}(t) \\ P_{0j}(t) = -\lambda_0 P_{0j}(t) - \lambda_0 P_{1j}(t) \end{cases}, \quad \text{with } P_{ij}(0) = \delta_{ij}$$

<u>Example</u> Linear growth with immigration.

Recall $\lambda_k = \lambda \cdot k + a$ ← immigration

↑ linear birth rate

$\mu_k = \mu \cdot k$

↑ linear death rate

Example: Linear growth with immigration.

Use forward equations to compute $E(X_t \mid X_0 = i)$

$$\begin{cases} P_{ij}'(t) = \lambda_{j-1} P_{i,j-1}(t) - (\lambda_j + \mu_j) P_{ij}(t) + \mu_{j+1} P_{i,j+1}(t) \\ P_{i0}'(t) = -\lambda_0 P_{i0}(t) + \mu_1 P_{i1}(t) \end{cases}$$

$E(X_t \mid X_0 = i) =$

$$P_{ij}'(t) = (\lambda(j-1) + a) P_{i,j-1}(t) - ((\lambda+\mu)j + a) P_{ij}(t) + \mu(j+1) P_{i,j+1}(t)$$

Example: Linear growth with immigration.

$$M'(t) =$$

$$=$$

$$=$$

$$\begin{cases} M'(t) = \\ M(0) = \end{cases}$$

$$M(t) = i + at \quad \text{if } \lambda = \mu$$

$$M(t) = \frac{a}{\lambda - \mu}\left(e^{(\lambda-\mu)t} - 1\right) + i e^{(\lambda-\mu)t} \quad \text{if } \lambda \neq \mu$$

Long run behavior of discrete time MC. Summary

Let $(X_n)_{n\geq 0}$ be a disrete time MC on $\{0, \dots, N\}$ with stationary transition probability matrix $P = (P_{ij})_{i,j=0}^{N}$.

- P is called regular if there exists k such that $[P^k]_{ij} > 0$ for all i,j. [P is regular iff (X_n) is irreducible and aperiodic]

<u>Thm</u>. If P is regular, then there exist $\pi_0, \dots, \pi_N \in \mathbb{R}$ s.t.

1) $\pi_i > 0 \quad \forall i$
2) $\sum_{i=0}^{N} \pi_i = 1$
3) $\forall j \quad \lim_{n\to\infty} P_{ij}^{(n)} = \pi_j$

$(\pi_0, \dots, \pi_n)$ is called limiting (stationary) distribution of (X_n)

$(\pi_0, \dots, \pi_n)$ is uniquely defined by the system of equations

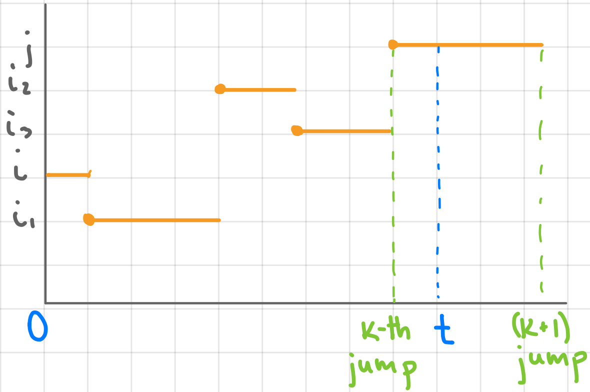
$$\begin{cases} \pi_j = \sum_{i=0}^{N} \pi_i P_{ij}, \\ \sum_{i=0}^{N} \pi_i = 1 \end{cases} \qquad (\pi_0, \pi_1, \dots, \pi_N) = (\pi_0, \dots, \pi_N) P$$

Long run behavior of continuous time MC.

Let $(X_t)_{t\geq 0}$ be a continuous time MC, $X_t \in \{0, \dots, N\}$
and let $(Y_n)_{n\geq 0}$ be the embedded jump chain.

Def. $(X_t)_{t\geq 0}$ is called irreducible if its jump chain $(Y_n)_{n\geq 0}$ is irreducible (consisting of one communicating class)

Thm. If $(X_t)_{t\geq 0}$ is irreducible, then



Idea of the proof:

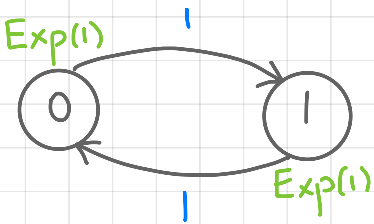
- Y_n is irreducible $\Rightarrow \exists\, i_1, \dots, i_{k-1}$ s.t.
  $P(Y_k = j, Y_{k-1} = i_{k-1}, \dots, Y_1 = i_1 \mid Y_0 = i) > 0$
- $P(k\text{-th jump} \leq t < (k+1)\text{-th jump}) > 0 \quad \forall t > 0$

Long run behavior of continuous time MC

Remarks: Continuous time MCs are "aperiodic"

All irreducible continuous time MCs are "regular"

Example.



Thm. If $(X_t)_{t\geq 0}$ is irreducible, then there exists $\pi_0, \dots, \pi_N$

1) $\pi_i > 0$, $\sum_{i=0}^{N} \pi_i = 1$

2) $\lim_{t\to\infty} P_{ij}(t) = \pi_j$ for all i

3) $\pi = (\pi_0, \dots, \pi_N)$ is uniquely determined by

π is called limiting/stationary/equilibrium distribution of (X_t)

Long run behavior of continuous time MC

Remark about 3) : $\pi Q = 0$ is equivalent to $\forall t$

($\Rightarrow$) If $\pi Q = 0$, then using Kolmogorov backward equation

$$(\pi P(t))' =$$

so $\pi P(t)$ is independent of t. Since $P(0) = I$, we get

$\forall t \qquad \pi P(t) =$

($\Leftarrow$) If $\pi P(t) = \pi$, then $(\pi P(t))' = 0$. Using Kolmogorov forward equation